A Level H2 Math

Functions Test 5

Q1

The function f is defined as follows.

$$\mathrm{f} : x \mapsto \sqrt{3}\sin x + \cos x, \quad x \in \mathbb{R}, \quad 0 < x < \pi.$$

(i) Write $\mathrm{f}(x)$ as $R\sin(x+\alpha)$, where R and α are constants with exact values to be found. [2]

(ii) Sketch the graph of $y = \mathrm{f}(x)$, stating the axial intercepts, and find the range of f. [3]

(iii) Hence, solve $\mathrm{f}(x) \le 1$ exactly. [2]

The function g is defined as follows:

$$\mathrm{g} : x \mapsto 2\cos\left(x + \frac{\pi}{6}\right), \quad x \in \mathbb{R}, \quad -\frac{\pi}{6} \le x \le b.$$

(iv) Write down the largest exact value of b, for g^{-1} to exist. [1]

(v) Taking the value of b found in part **(iv)**, show that the composite function $\mathrm{g}^{-1}\mathrm{f}$ exists and solve $\mathrm{g}^{-1}\mathrm{f}(x) = x$ exactly. [3]

Q2

It is given that

$$\mathrm{f}(x) = \begin{cases} b\sqrt{1 - \dfrac{x^2}{a^2}} & \text{for } -a < x \le a \\ -a\sqrt{1 - \dfrac{(x-2a)^2}{a^2}} & \text{for } a < x \le 3a \end{cases}$$

and that $\mathrm{f}(x+4a) = \mathrm{f}(x)$ for all real values of x, where a and b are real constants and $0 < a < b$.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-a \le x \le 8a$. **[3]**

(ii) Use the substitution $x = a\cos\theta$ to find the exact value of $\int_{3a}^{4a} \mathrm{f}(x)\,\mathrm{d}x$ in terms of a and π. **[5]**

Q3

The function f is defined by $\mathrm{f}: x \mapsto \dfrac{1}{x^2-1}, \quad x \in \mathbb{R}, \quad x > 1$.

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}. [3]

(ii) On the same diagram, sketch the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$ stating the equations of any asymptotes and showing the relationships between the graphs clearly. [4]

(iii) State the set of values of x such that $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$. [1]

Answers

Functions Test 5

Q1

(i) $f(x) = R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

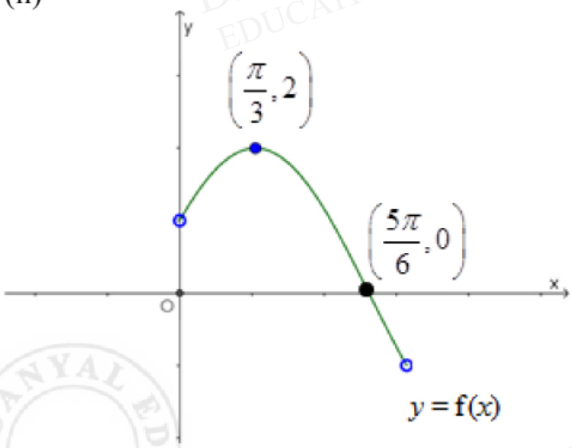
compare with $f(x) = \sqrt{3}\sin x + \cos x$

$\Rightarrow R\cos\alpha = \sqrt{3},\ R\sin\alpha = 1$

$\Rightarrow R = \sqrt{1^2 + \sqrt{3}^2} = 2, \quad \alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

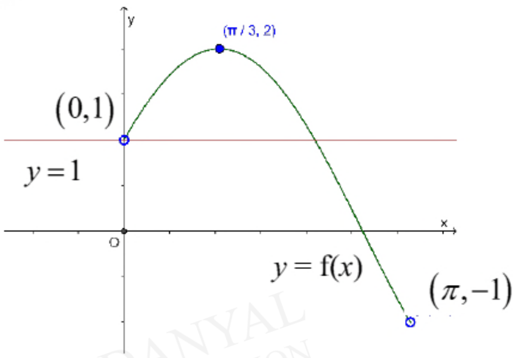
$f(x) = 2\sin\left(x + \frac{\pi}{6}\right)$

(ii)



The range of f is $(-1, 2]$.

(iii)



$f(x) = 2\sin(x + \frac{\pi}{6}) = 1$

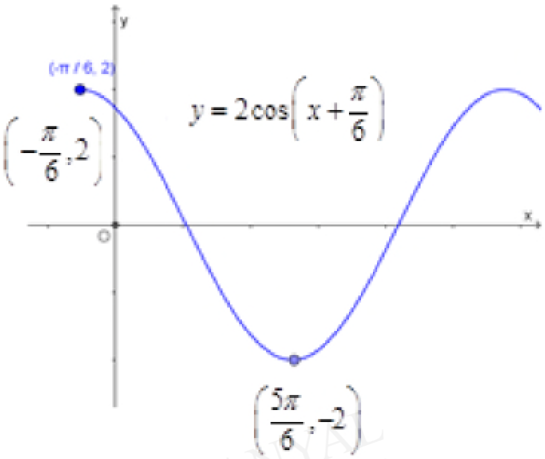
$\Rightarrow x + \frac{\pi}{6} = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6}$

$\Rightarrow x = 0$ or $\frac{2\pi}{3}$

The set of values of x is $\left[\frac{2\pi}{3}, \pi\right)$

(iv)

$g(x) = 2\cos\left(x + \frac{\pi}{6}\right), \quad -\frac{\pi}{6} \le x \le b.$



For g^{-1} to exist, g has to be a 1-1 function.

The largest exact value of b, is $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

(v)

The domain of g^{-1} = the range of $g = [-2, 2]$.

Range of $f = (-1, 2] \subseteq$ Domain of $g^{-1} = [-2, 2]$, therefore $g^{-1}f$ exists.

$$g^{-1}f(x) = x, \quad 0 < x < \pi$$

$$g(x) = f(x)$$

$$2\cos(x + \frac{\pi}{6}) = 2\sin(x + \frac{\pi}{6})$$

$$\tan(x + \frac{\pi}{6}) = 1$$

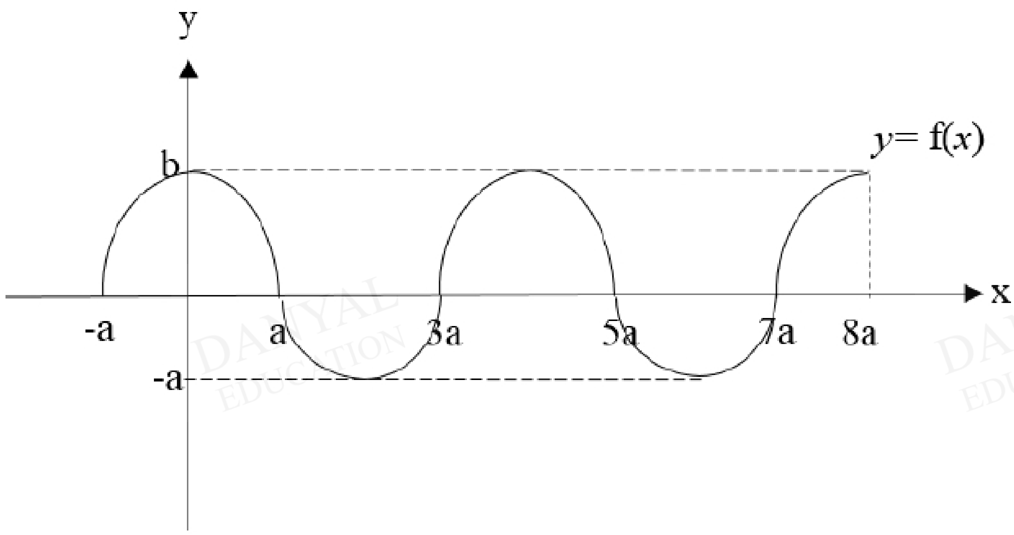
$$x + \frac{\pi}{6} = \frac{\pi}{4}$$

$$x = \frac{\pi}{12}$$

(Note: $0 < x \leq \frac{5\pi}{6}$ considering domain of f and g)

Q2

(i)



(ii)

$$\int_{3a}^{4a} \mathrm{f}(x)\,\mathrm{d}x$$

$$=\int_{-a}^{0} b\sqrt{1-\frac{x^2}{a^2}}\ \mathrm{d}x$$

$$=b\int_{\pi}^{\frac{\pi}{2}} \sqrt{1-\frac{a^2\cos^2\theta}{a^2}}\ (-a\sin\theta)\ \mathrm{d}\theta$$

$$=ab\int_{\frac{\pi}{2}}^{\pi} \sin^2\theta\ \mathrm{d}\theta$$

$$=ab\int_{\frac{\pi}{2}}^{\pi} \frac{1-\cos 2\theta}{2}\ \mathrm{d}\theta$$

$$=\frac{ab}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$=\frac{ab}{2}\left[\pi-\frac{\pi}{2}\right]$$

$$=\frac{\pi}{4}ab$$

Q3

(i)

$\mathrm{f}: x \mapsto \dfrac{1}{x^2-1}$

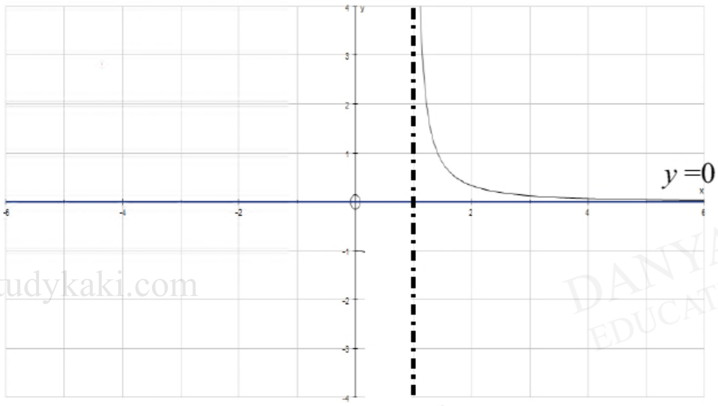
Let $y = \dfrac{1}{x^2-1}$

$x^2 = \dfrac{1}{y} + 1$

$x = \pm\sqrt{1+\dfrac{1}{y}}$

Since $x > 1$, $x = \sqrt{1+\dfrac{1}{y}}$

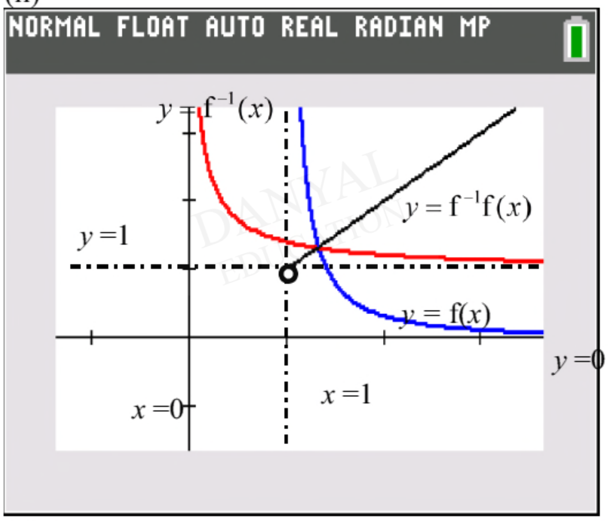
$\mathrm{f}^{-1}(x) = \sqrt{1+\dfrac{1}{x}} = \sqrt{\dfrac{1+x}{x}}$.



From graph of f, $\mathrm{R_f} = (0, \infty)$

$\therefore \mathrm{D_{f^{-1}}}(x) = (0, \infty)$.

(ii)



(iii)

Since $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x$ have the same rule, we investigate the domain

$D_{\mathrm{f^{-1}f}} = (1, \infty)$ $D_{\mathrm{ff^{-1}}} = (0, \infty)$

Taking the intersection of these domains,

Range of values is $x > 1$.