A Level H2 Math

Functions Test 4

Q1

The function f is defined by

$$\mathrm{f}: x \mapsto \frac{\mathrm{e}^{x}-1}{\mathrm{e}-1} \quad \text{for} \quad x \in \mathbb{R}.$$

Sketch the graph of $y=\mathrm{f}(x)$ and state the range of f. [3]

Another function h is defined by

$$\mathrm{h}: x \mapsto \begin{cases} (x-1)^{2}+1 & \text{for} \quad x \leq 1 \\ 1-\dfrac{|1-x|}{2} & \text{for} \quad 1<x \leq 4 \end{cases}$$

Sketch the graph of $y=\mathrm{h}(x)$ for $x \leq 4$ and explain why the composite function $\mathrm{f}^{-1}\mathrm{h}$ exists. Hence find the exact value of $\left(\mathrm{f}^{-1}\mathrm{h}\right)^{-1}(3)$. [7]

Q2

The function f is defined by

$$\mathrm{f}: x \mapsto \frac{\pi}{2}\tan\left(\frac{x}{2}\right), \quad x \in \mathbb{R}, -2\pi \leq x \leq 2\pi.$$

(i) Explain why f^{-1} does not exist. [2]

(ii) The domain of f is restricted to $(-\pi, a)$ such that a is the largest value for which the inverse function f^{-1} exists. State the exact value of a and define f^{-1} in a similar form. [3]

In the rest of the question, the domain of f is $(-\pi, a)$, where a takes the value found in part **(ii)**.

(iii) Sketch, in a single diagram, the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$, labelling each graph clearly. Write down the equation of the line in which the graph of $y=\mathrm{f}(x)$ must be reflected in order to obtain the graph of $y=\mathrm{f}^{-1}(x)$ and draw this line on your diagram. [3]

(iv) Verify that $x=\frac{\pi}{2}$ is a root of the equation $x=\mathrm{f}(x)$. Hence, explain why $x=\frac{\pi}{2}$ is also a solution to the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. [2]

Q3

The function f is given by $f : x \mapsto 3 + \frac{1}{x-2}$ for $x \in \square$, $x > 2$.

(i) Find $f^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Explain why the composite function f^2 exists. [1]

(iii) Find the value of x for which $f^2(x) = x$. Explain why this value of x satisfies the equation $f(x) = f^{-1}(x)$. [3]

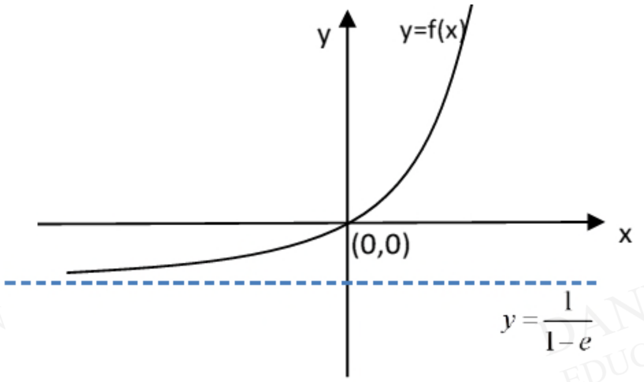
Answers

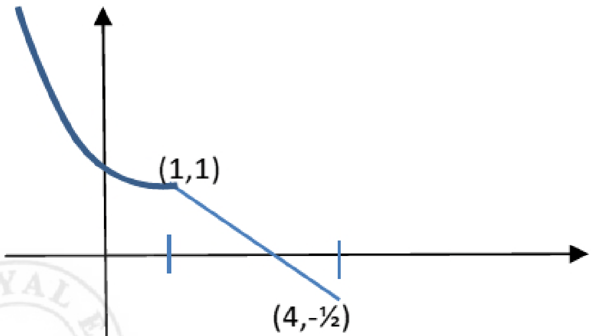
Functions Test 4

Q1

Soln:

$R_f = \left(\frac{1}{1-e}, \infty\right)$





$R_h = \left[-\frac{1}{2}, \infty\right)$

$D_{f^{-1}} = R_f = \left(\frac{1}{1-e}, \infty\right) = (-0.582, \infty)$

Hence $R_h \subseteq D_{f^{-1}}$, so $f^{-1}h$ exists.

Let $\left(f^{-1}h\right)^{-1}(3) = k$

$\Rightarrow f^{-1}h(k) = 3$

$\Rightarrow h(k) = f(3)$

$\Rightarrow h(k) = \frac{e^3 - 1}{e - 1} = e^2 + e + 1$

Since $e^2+e+1 > 1$,
hence $h(x) = (x-1)^2+1$

$(k-1)^2 + 1 = e^2 + e + 1$

$\Rightarrow k = 1 \pm \sqrt{e^2 + e}$

Alternative method: use $f^{-1}(x)$

Let $\left(f^{-1}h\right)^{-1}(3) = k$

$\Rightarrow f^{-1}h(k) = 3$

$\Rightarrow \ln\left[1 + (e-1)h(k)\right] = 3$

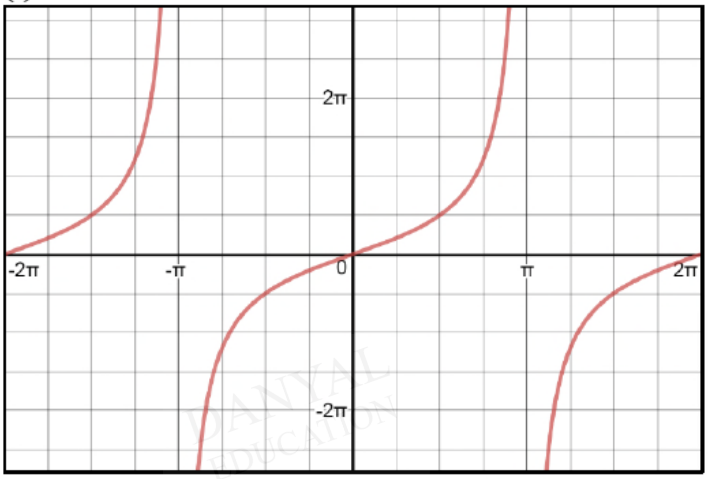
$\Rightarrow 1 + (e-1)h(k) = e^3$

$\Rightarrow h(k) = \frac{e^3 - 1}{e - 1} = e^2 + e + 1$

Since $x < 1$, hence the exact value of $\left(f^{-1}h\right)^{-1}(3) = 1 - \sqrt{e^2 + e}$.

Q2

(i)



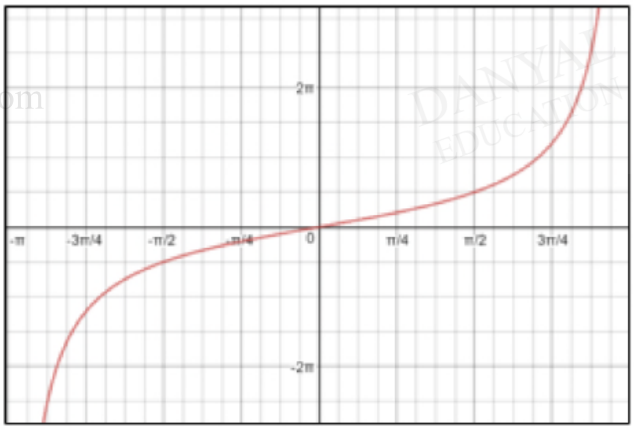
$y = f(x)$

The horizontal line $y = 1$ cuts the graph of $y = f(x)$ at **2 points**. Thus, f(x) is not a one-one function and the inverse of $f(x)$ does not exist for the domain $[-2\pi, 2\pi\ ]$.

OR

Any horizontal line $y = k$ $(k \in \mathbb{R})$ cuts the graph at more than one point. Thus, $f(x)$ is not a one-one function and the inverse of $f(x)$ does not exist for the domain $[-2\pi, 2\pi\ ]$.

(ii)



$a = \pi$

To make x the subject of y

$$y = \frac{\pi}{2}\tan\left(\frac{x}{2}\right)$$

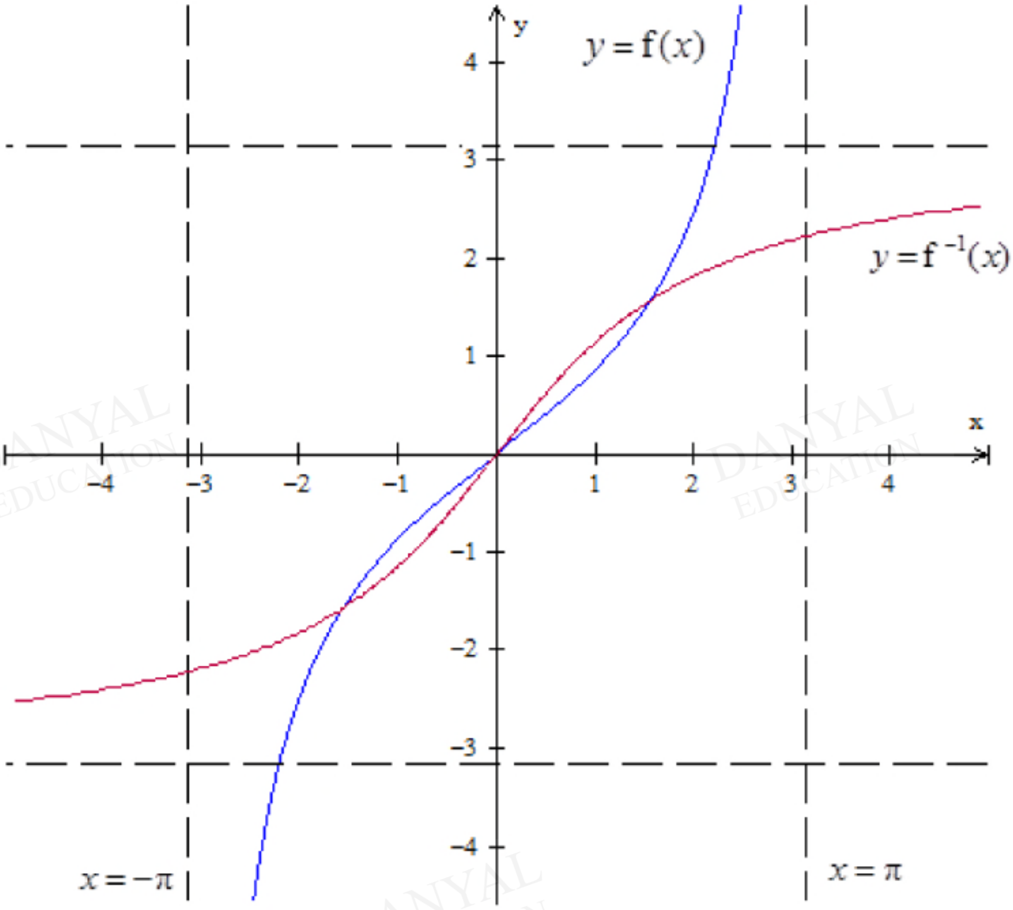
$$\frac{2y}{\pi} = \tan\left(\frac{x}{2}\right)$$

$$\tan^{-1}\left(\frac{2y}{\pi}\right) = \frac{x}{2}$$

$$\Rightarrow x = 2\tan^{-1}\left(\frac{2y}{\pi}\right)$$

$$f^{-1} : x \mapsto 2\tan^{-1}\left(\frac{2x}{\pi}\right), \quad x \in R.$$

(iii)



The line required is $y=x$.

(iv)

$$\mathrm{f}\left(\frac{\pi}{2}\right)=\frac{\pi}{2}\tan\left(\frac{\pi}{4}\right)=\frac{\pi}{2}$$

Thus, $x=\frac{\pi}{2}$ is a root of the equation $x=\mathrm{f}(x)$.

Since the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ intersect along the line $y=x$, and since $x=\frac{\pi}{2}$ is a root of the equation $x=\mathrm{f}(x)$, thus, the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ must also intersect at the point $x=\frac{\pi}{2}$.

Q3

(i)

$$\mathrm{f}: x \mapsto 3+\frac{1}{x-2},\ x \in \mathbb{R},\ x>2$$

Let $y=\mathrm{f}(x)$.

$$y=3+\frac{1}{x-2}$$

$$x-2=\frac{1}{y-3}$$

$$x=2+\frac{1}{y-3}$$

$$\therefore\ \mathrm{f}^{-1}(x)=2+\frac{1}{x-3},\ x \in \mathbb{R},\ x>3$$

(ii)

$$\mathrm{D_f}=(2,\infty)$$

$$\mathrm{R_f}=(3,\infty)$$

Since $\mathrm{R_f} \subseteq \mathrm{D_f}$, the composite function f^2 exists.

(iii)

$$\mathrm{f}^2(x)=x$$

$$\mathrm{f}\left(3+\frac{1}{x-2}\right)=x$$

$$3+\frac{1}{3+\frac{1}{x-2}-2}=x$$

$$3+\frac{1}{\left(\frac{x-1}{x-2}\right)}=x$$

$$\frac{3(x-1)+(x-2)}{x-1}=x$$

$$4x-5=x(x-1)$$

$$x^2-5x+5=0$$

Using GC, $x=1.38$ (rej $\because 1.38 \notin \mathrm{D_f}$) or $x=3.62$

$$\mathrm{ff}(x)=x$$

$$\mathrm{f}^{-1}\mathrm{ff}(x)=\mathrm{f}^{-1}(x)$$

$$\mathrm{f}(x)=\mathrm{f}^{-1}(x)$$

Therefore $x=3.62$ satisfies $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$.