A Level H2 Math

Functions Test 3

Q1

The *floor function*, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x. For example, $\lfloor -2.1 \rfloor = -3$ and $\lfloor 3.5 \rfloor = 3$.

The function f is defined by

$$\mathrm{f}(x) = \begin{cases} \lfloor x \rfloor & \text{for } x \in \mathbb{R}, \quad -1 \le x < 2, \\ 0 & \text{for } x \in \mathbb{R}, \quad 2 \le x < 3, \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

It is given that $\mathrm{f}(x) = \mathrm{f}(x+4)$.

(i) Find the values of $\mathrm{f}(-1.2)$ and $\mathrm{f}(3.6)$. [2]

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-2 \le x < 4$. [2]

(iii) Hence evaluate $\int_{-2}^{4} \mathrm{f}(x)\, \mathrm{d}x$. [1]

Q2

The function f is defined by

$$f : x \mapsto \sqrt{3}\sin x + \cos x\,, \qquad x \in \mathbb{R}\,,\ -\pi < x < \tfrac{\pi}{6}\,.$$

(i) Express f in the form $R\sin(x+\alpha)$, where R and α are exact constants to be determined, $R > 0$, $0 \le \alpha \le \frac{\pi}{2}$. [2]

(ii) Sketch f, giving the exact coordinates of the turning point and the end-points. Deduce the exact range of f. [4]

(iii) The function g is defined by

$$g : x \mapsto \frac{1}{2} - |x-1|\ , \qquad x \in \mathbb{R},\ -\frac{5}{2} \le x \le \frac{1}{2}.$$

Explain why the composite function fg exists. Find the range of fg . [3]

(iv) The domain of f is restricted such that the function f^{-1} exists. Find the largest domain of f for which f^{-1} exists. Define f^{-1} in a similar form. [4]

Q3

(i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$ where R and α are exact positive constants to be found. [2]

(ii) State a sequence of transformations which transform the graph of $y = \cos x$ to the graph of $y = \sqrt{3}\cos x - \sin x$. [2]

The function f is defined by $\mathrm{f} : x \mapsto \sqrt{3}\cos x - \sin x,\ 0 \le x \le 2\pi$.

(iii) Sketch the graph of $y = \mathrm{f}(x)$ and state the range of f. [3]

The function g is defined by $\mathrm{g} : x \mapsto \mathrm{f}(x),\ 0 \le x \le k$.

(iv) Given that g^{-1} exists, state the largest exact value of k and find $\mathrm{g}^{-1}(x)$. [3]

The function h is defined by $\mathrm{h} : x \mapsto x - 2,\ x \ge 0$.

(v) Explain why the composite function fh does not exist. [1]

Answers

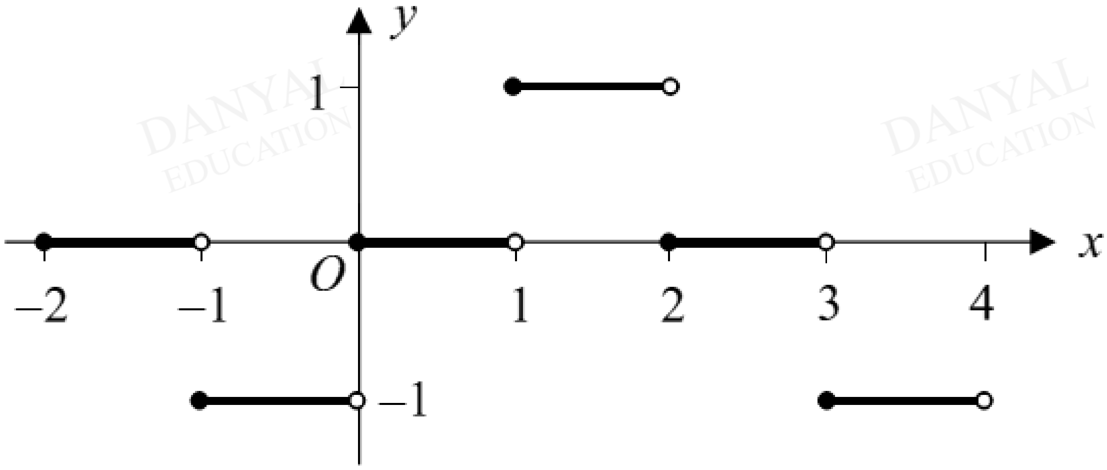
Functions Test 3

Q1

(i) $f(-1.2) = f(2.8) = 0$

$f(3.6) = f(-0.4) = -1$

(ii)



(iii) $\int_{-2}^{4} f(x)\,dx = -1 + 1 - 1 = -1$

Q2

(i)

$f(x) = \sqrt{3}\sin x + \cos x$

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\cos\alpha = \sqrt{3}$ ...(1)

$R\sin\alpha = 1$ ...(2)

$(1)^2 + (2)^2$,

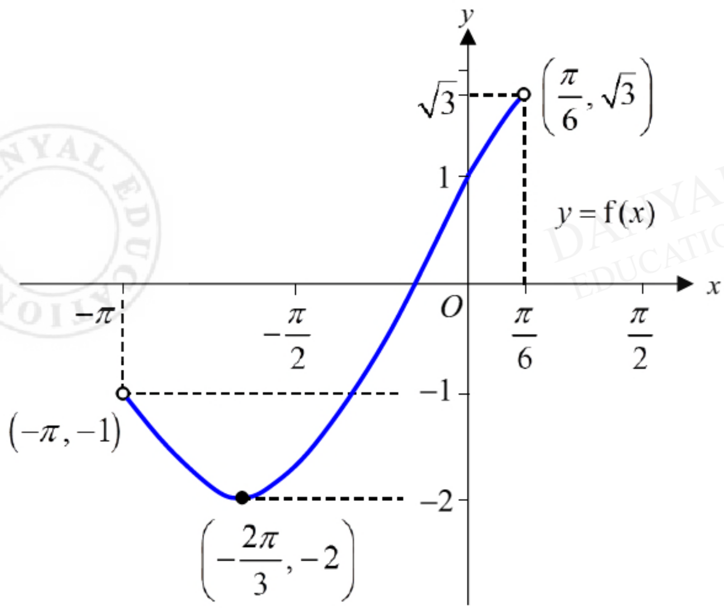
$\therefore\ R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

(1) / (2), $\tan\alpha = \dfrac{1}{\sqrt{3}}$,

$\therefore\ \alpha = \dfrac{\pi}{6}$

Hence $f(x) = 2\sin\left(x + \dfrac{\pi}{6}\right)$

(ii)



When $y = -2$,

$2\sin(x + \frac{\pi}{6}) = -2$

$\sin(x + \frac{\pi}{6}) = -1$

$x + \frac{\pi}{6} = -\frac{\pi}{2} \Rightarrow x = -\frac{2\pi}{3}$

$\therefore$ turning point is $\left(-\dfrac{2\pi}{3}, -2\right)$.

$R_f = [-2, \sqrt{3})$

(iii)

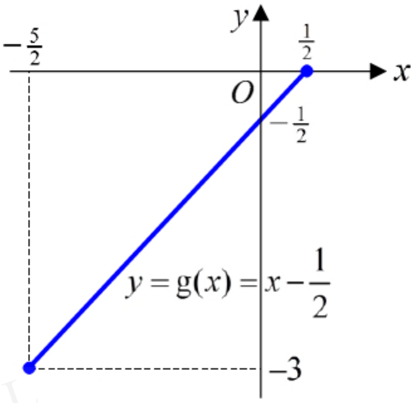
Since $-\frac{5}{2} \le x \le \frac{1}{2}$,

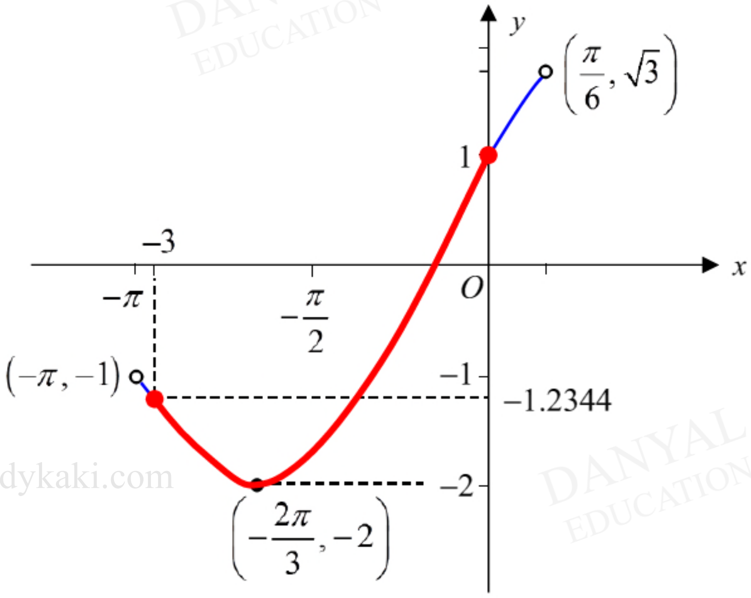
$g(x) = \frac{1}{2} + x - 1 = x - \frac{1}{2}$

$R_g = [-3, 0]$

$D_f = (-\pi, \frac{\pi}{6})$

Since $R_g \subset D_f$, fg exists





$$\overbrace{[-\tfrac{5}{2}, \tfrac{1}{2}]}^{D_g} \rightarrow \overset{R_g = \text{restricted } D_f}{[-3, 0]} \rightarrow \overset{\text{restricted } R_f = R_{fg}}{[-2, 1]}$$

$\therefore\ R_{fg} = [-2, 1]$

(iv)

From the graph,

largest domain for f $= [-\frac{2\pi}{3}, \frac{\pi}{6})$

Let $y = 2\sin(x + \frac{\pi}{6})$

$x = \sin^{-1}\left(\frac{y}{2}\right) - \frac{\pi}{6}$

$f^{-1} : x \mapsto \sin^{-1}\left(\frac{x}{2}\right) - \frac{\pi}{6}$, $x \in \mathbb{R}$, $-2 \le x < \sqrt{3}$.

Q3

(i) $\sqrt{3}\cos x - \sin x = R\cos(x+\alpha)$

$$R = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{4} = \underline{\underline{2}}$$

$$\alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \underline{\underline{\frac{\pi}{6}}}$$

(ii) $y = \sqrt{3}\cos x - \sin x = 2\cos\left(x + \frac{\pi}{6}\right)$

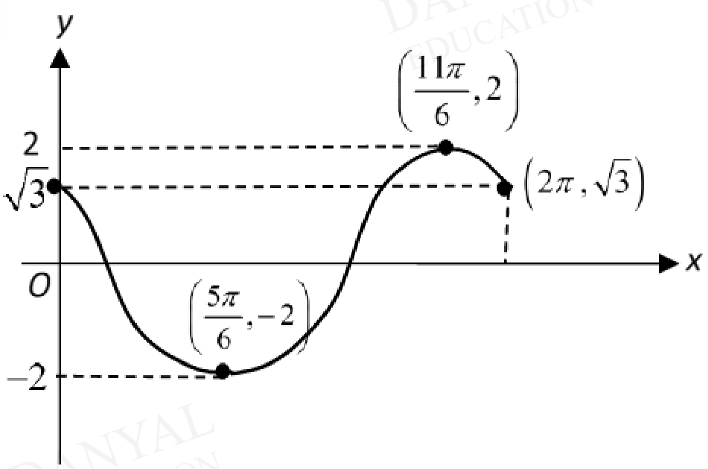
$$y = \cos x \xrightarrow{A} y = \cos(x+\alpha) \xrightarrow{B} y = R\cos(x+\alpha)$$

A: Translation by α radians in the negative x-direction, followed by

B: Scaling parallel to the y-axis by a scale factor R.

[can be B followed by A]

(iii) $f : x \mapsto \sqrt{3}\cos x - \sin x,\ 0 \le x \le 2\pi$



Range of f, $R_f = \underline{\underline{[-2, 2]}}$.

(iv) $g : x \mapsto f(x),\ 0 \le x \le k.$

Largest $k = \dfrac{5\pi}{6}$.

$$
\begin{aligned}
\text{Let} \quad y &= g(x). \\
y &= 2\cos\left(x + \frac{\pi}{6}\right) \\
\cos\left(x + \frac{\pi}{6}\right) &= \frac{y}{2} \\
\Rightarrow x &= \cos^{-1}\frac{y}{2} - \frac{\pi}{6} \\
\therefore g^{-1}(x) &= \cos^{-1}\frac{x}{2} - \frac{\pi}{6}
\end{aligned}
$$

(v) $h : x \mapsto x - 2,\ x \ge 0$

Since $R_h = [-2, +\infty)$ and $D_f = [0, 2\pi]$,

$R_h \not\subset D_f$, fh does not exist.