A Level H2 Math

Functions Test 2

Q1
The function f is defined by

$$\mathrm{f}: x \mapsto \frac{1}{3}\tan\left(\frac{x}{3}\right) \text{ for } x \in \mathbb{R},\ 0 \le x < \frac{3\pi}{2}.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$, indicating clearly the vertical asymptote. [2]

(ii) State the equation of the line of reflection between the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, and hence sketch the graph of $y = \mathrm{f}^{-1}(x)$ on the same diagram, indicating clearly the horizontal asymptote. [2]

The solutions to the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ are $x = 0$ and $x = \alpha$, where $0 < \alpha < \frac{3\pi}{2}$.

(iii) Using the diagram drawn, find, in terms of α, the area of the region bounded by the curves $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$. [5]

Another function g is defined by

$$\mathrm{g}: x \mapsto \mathrm{e}^{x} \text{ for } x \in \mathbb{R},\ x \ge -2.$$

(iv) Show that the composite function gf exists and define gf in a similar form. [3]

Q2

A function f is said to self-inverse if $f(x) = f^{-1}(x)$ for all x in the domain of f.

The functions f and g are defined by

$$f : x \mapsto \frac{7-3x}{3-x}, \qquad x \in \mathbb{R}, x \neq 3,$$
$$g : x \mapsto \left|(2-x)(1+x)\right|, \qquad x \in \mathbb{R}, x \in (-\infty, -1].$$

(i) Explain why f^{-1} exists and show that f is self-inverse. Hence, or otherwise, evaluate $f^{2003}(5)$. [4]

(ii) Find an expression for $g^{-1}(x)$. [3]

(iii) Sketch, on the same diagram, the graphs of $y = g(x)$ and $y = g^{-1}(x)$, illustrating clearly the relationship between the two graphs, and labelling the axial intercept(s), if any. Write down the set of values of x that satisfies the equation $gg^{-1}(x) = x$. [3]

(iv) Show that fg^{-1} exists. Find the exact range of fg^{-1}. [3]

Q3

(i) Express $\sin x + \sqrt{3}\cos x$ as $R\sin(x+\alpha)$, where $R > 0$ and α is an acute angle. [1]

The function f is defined by

$$\mathrm{f} : x \mapsto \sin x + \sqrt{3}\cos x, \quad x \in \mathbb{R},\ -\tfrac{1}{3}\pi \le x \le \tfrac{1}{6}\pi.$$

(ii) Sketch the graph of $y = \mathrm{f}(x)$. [2]

(iii) Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. On the same diagram as in part **(ii)**, sketch the graph of $y = \mathrm{f}^{-1}(x)$, indicating the equation of the line of symmetry. [4]

(iv) Using integration, find the area of the region bounded by the graph of f^{-1} and the axes. [3]

The function g is defined by

$$\mathrm{g} : x \mapsto \left|\ln(x+2)\right|, \text{ for } x \in \mathbb{R},\ x > -2.$$

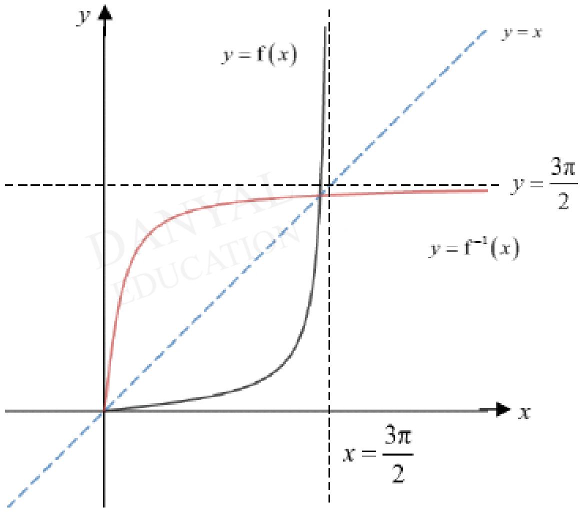
(v) Show that the composite function gf^{-1} exists, and find the range of gf^{-1}. [3]

Answers

Functions Test 2

Q1

(i)



Many students did not fully extend the curve past $x=\frac{3\pi}{2}$ and/or $y=\frac{3\pi}{2}$

(ii) $y=x$

(iii) **Method ①:**

$$\text{Area } = 2\int_0^{\alpha} x-\frac{1}{3}\tan\left(\frac{x}{3}\right)\mathrm{d}x$$

$$=2\left[\frac{x^2}{2}-\ln\left|\sec\left(\frac{x}{3}\right)\right|\right]_0^{\alpha}$$

$$=2\left[\frac{\alpha^2}{2}-\ln\left|\sec\left(\frac{\alpha}{3}\right)\right|-0+0\right]$$

$$=\alpha^2+2\ln\left[\cos\left(\frac{\alpha}{3}\right)\right]$$

Most students did not use this method, opting for the more tedious alternative.
Students can use area of triangle formula $\frac{1}{2}\alpha(\alpha)$ instead of $\int_0^{\alpha} x\ \mathrm{d}x$

(Some used $\frac{1}{2}\alpha\left(\frac{1}{3}\tan\left(\frac{\alpha}{3}\right)\right)$ which is not simplified

Method ②:

$$\text{Area } = \int_0^{\alpha} 3\tan^{-1}3x - \frac{1}{3}\tan\left(\frac{x}{3}\right)\mathrm{d}x$$

$$= \left[3x\tan^{-1}3x\right]_0^{\alpha} - \int_0^{\alpha} 3x\frac{3}{1+(3x)^2}\,\mathrm{d}x - \left[\ln\left(\sec\frac{x}{3}\right)\right]_0^{\alpha}$$

$$= 3\alpha\tan^{-1}3\alpha - \frac{1}{2}\int_0^{\alpha}\frac{18x}{1+9x^2}\,\mathrm{d}x - \ln\left(\sec\frac{\alpha}{3}\right) + \ln 1$$

$$= 3\alpha\tan^{-1}3\alpha - \frac{1}{2}\left[\ln\left(1+9x^2\right)\right]_0^{\alpha} - \ln\left(\sec\frac{\alpha}{3}\right)$$

$$= 3\alpha\tan^{-1}3\alpha - \frac{1}{2}\ln\left(1+9x^2\right) - \ln\left(\sec\frac{\alpha}{3}\right)$$

Note the two answers are equal. Most students did this method, not utilizing the symmetry of the curves.

(iv)

$R_f = [0, \infty)$

$D_g = [-2, \infty)$

Since $R_f \subseteq D_g$, gf exists.

$$\mathrm{gf}(x) = \mathrm{g}\left[\frac{1}{3}\tan\left(\frac{x}{3}\right)\right]$$

$$= \mathrm{e}^{\frac{1}{3}\tan\left(\frac{x}{3}\right)}$$

$$D_{gf} = D_f = \left[0, \frac{3\pi}{2}\right)$$

$$\mathrm{gf}: x \mapsto \mathrm{e}^{\frac{1}{3}\tan\left(\frac{x}{3}\right)} \text{ for } x\in\mathbb{R},\ 0\le x<\frac{3\pi}{2}.$$

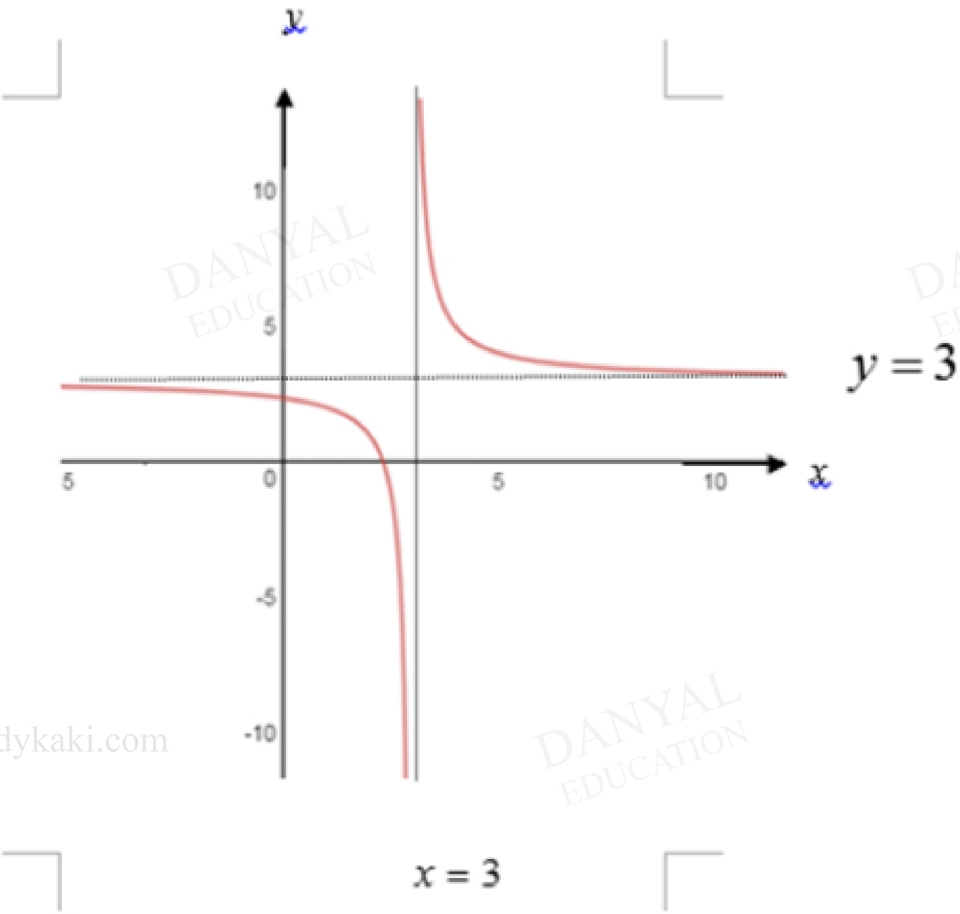
Common mistakes:

$D_{gf} = D_g = [-2, \infty)$

Many students did not put in similar form

Q2

(i)



Since any horizontal line $y = a, a \in \mathbb{R}$, intersects the graph of $y = \mathrm{f}(x)$ at most once, the function f is one-one. It follows that f^{-1} exists.

OR

Since any horizontal line $y = a, a \in \mathrm{R}_f$, intersects the graph of $y = \mathrm{f}(x)$ exactly once, the function f is one-one. It follows that f^{-1} exists.

Let $y = \dfrac{7-3x}{3-x}$

$y(3-x) = 7 - x$

$x = \dfrac{7-3y}{3-y}$

Since $\mathrm{f}^{-1}(x) = \dfrac{7-3x}{3-x}$, $x \in \mathbb{R}, x \neq 3$,

$\therefore \mathrm{f}^{-1} = \mathrm{f}$. (shown)

$D_{\mathrm{f}^{-1}} = R_{\mathrm{f}} = (-\infty, 3) \cup (3, \infty) = D_{\mathrm{f}}$

Note that $\mathrm{f}^{-1}\mathrm{f}(x) = x$. Therefore, $\mathrm{f}^{2003}(5) = \underbrace{\mathrm{fff...f}}_{2003\text{ times}}(5) = \mathrm{f}\left(\underbrace{\mathrm{f}^{-1}\mathrm{f......f}^{-1}\mathrm{f}}_{1000\text{ times of } \mathrm{f}^{-1}\mathrm{f}}(5)\right) = \mathrm{f}(5) = 4.$

(ii)

$$|(2-x)(1+x)| = \begin{cases} (2-x)(1+x), & -1 \le x \le 2, \\ -(2-x)(1+x), & x < -1 \text{ or } x > 2. \end{cases}$$

For $x \in (-\infty, -1]$, $y = -(2-x)(1+x)$

Method 1

$x^2 - x - 2 - y = 0$

$x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2-y)}}{2(1)}$

$x = \dfrac{1 \pm \sqrt{9+4y}}{2}$

Method 2

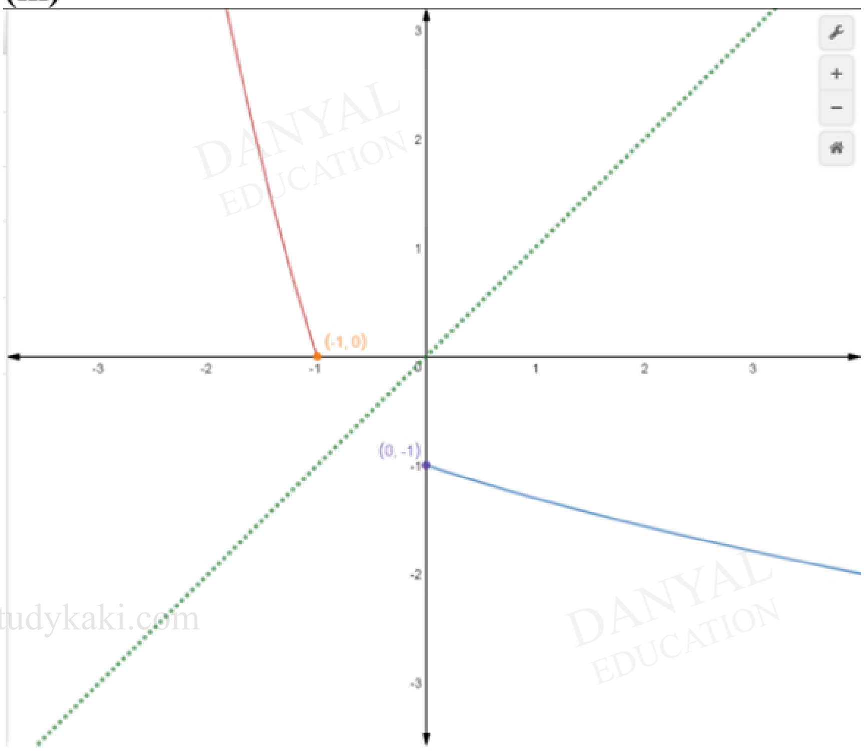
$y = x^2 - x - 2 = (x - 0.5)^2 - 2.25$

$x = 0.5 \pm \sqrt{y + 2.25}$

$x=\dfrac{1+\sqrt{9+4y}}{2}$ (rejectd $\because x\le -1$) or $\dfrac{1-\sqrt{9+4y}}{2}$

$\therefore \mathrm{g}^{-1}(x)=\dfrac{1}{2}-\sqrt{x+\dfrac{9}{4}}$

(iii)



For $\mathrm{g}\,\mathrm{g}^{-1}(x)=x$,

$D_{\mathrm{g}\mathrm{g}^{-1}}=D_{\mathrm{g}^{-1}}$.

$\therefore x\in[0,\infty)$ or $x\ge 0$

(iv)

Since $R_{\mathrm{g}^{-1}}=(-\infty,-1]$ and $\mathrm{D_f}=(-\infty,\infty)\setminus\{3\}$

$R_{g^{-1}}\subseteq D_f$.

$\therefore \mathrm{fg}^{-1}$ exists.

Using the graph of $y=\mathrm{g}^{-1}(x)$ in part **(ii)**, $R_{\mathrm{g}^{-1}}=(-\infty,-1]$.

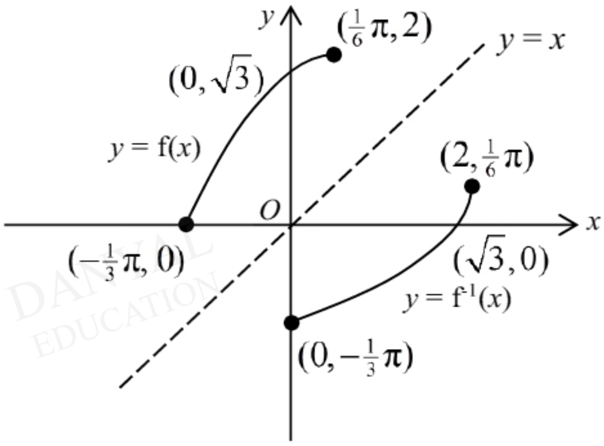
From graph of $y=\mathrm{f}(x)$ in **(i)** in $(-\infty,-1]$.

$\therefore R_{\mathrm{fg}^{-1}}=[2.5,3)$

Q3

(i) Using R formula, $\sin x + \sqrt{3}\cos x = 2\sin(x + \frac{1}{3}\pi)$

(ii)



(iii) To find f^{-1}:

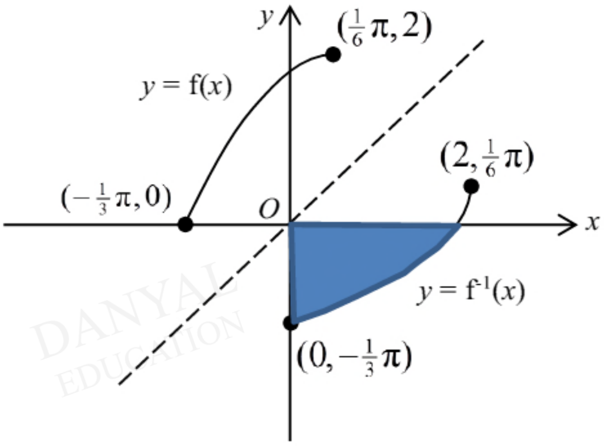
Let $y = 2\sin(x + \frac{1}{3}\pi)$

$\therefore x = -\frac{1}{3}\pi + \sin^{-1}(\frac{1}{2}y)$

$f^{-1}(x) = -\frac{1}{3}\pi + \sin^{-1}(\frac{1}{2}x)$

$D_{f^{-1}} = R_f = [0, 2]$

(iv) For the area bounded by the graph of f^{-1} and the axes:



By symmetry,

Area

$$= \int_{-\frac{\pi}{3}}^{0} f(x)\,dx = \int_{-\frac{\pi}{3}}^{0} (\sin x + \sqrt{3}\cos x)\,dx$$

$$= \left[-\cos x + \sqrt{3}\sin x\right]_{-\frac{\pi}{3}}^{0} = (-1 + 0) - \left(-\frac{1}{2} - \frac{3}{2}\right) = 1$$

(v) gf^{-1} exists if $R_{f^{-1}} \subseteq D_g$.

Since

$R_{f^{-1}} = [-\frac{1}{3}\pi, \frac{1}{6}\pi]$

$D_g = (-2, \infty)$,

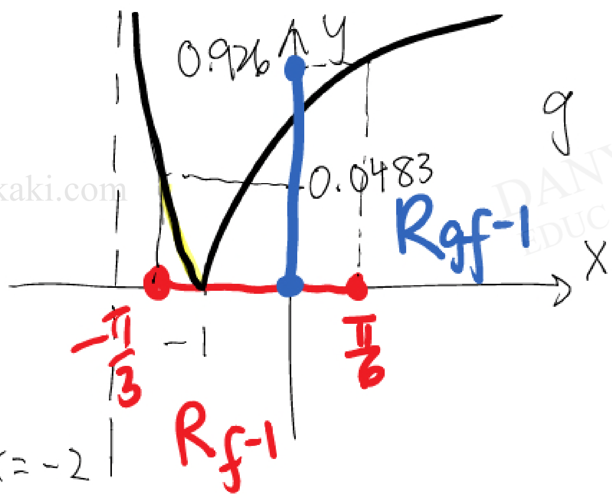
Ie. $R_{f^{-1}} \subseteq D_g \Rightarrow gf^{-1}$ exists

To find the range of gf^{-1}:

Method 1 (two stage mapping method)

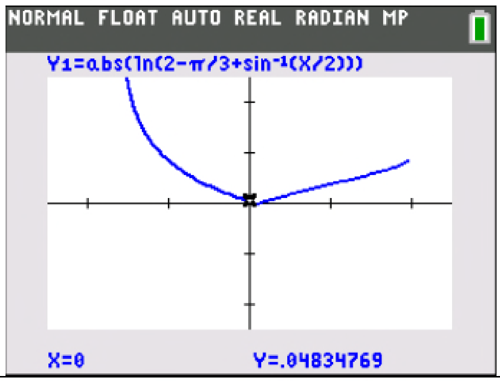
$$D_{f^{-1}} \xrightarrow{f^{-1}} R_{f^{-1}} \xrightarrow{g} R_{gf^{-1}}$$

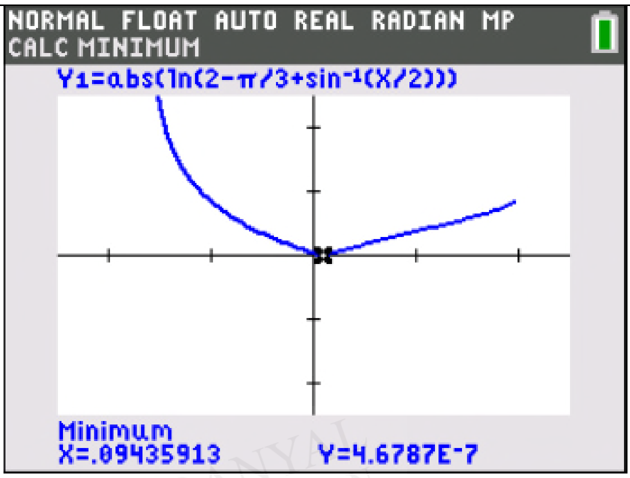
$[0, 2]$ $\quad$ $[-\frac{1}{3}\pi, \frac{1}{6}\pi]$ $\quad$?

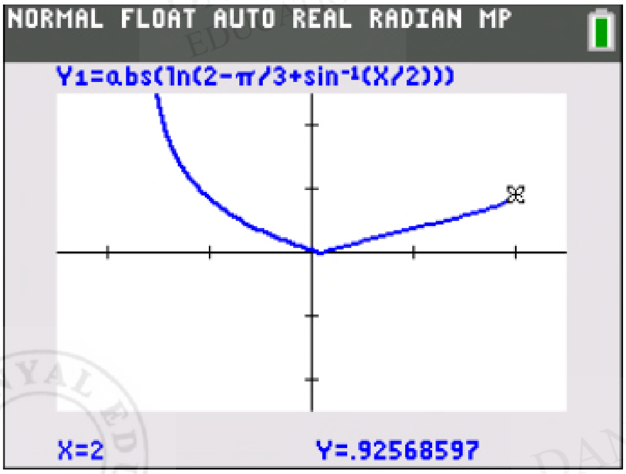


$R_{gf^{-1}} = [0,\ 0.926]$

Method 2 (find gf^{-1}) (need to use GC to see shape)







$$gf^{-1}(x) = \left|\ln(2 - \tfrac{1}{3}\pi + \sin^{-1}(\tfrac{1}{2}x))\right|$$

$$D_{gf^{-1}} = D_{f^{-1}} = [0,2]$$

$$R_{gf^{-1}} = [0, 0.926]$$