A Level H2 Math

Functions Test 1

Q1

(a) By writing

$$\sin\left(x+\frac{1}{4}\right)\pi-\sin\left(x-\frac{3}{4}\right)\pi$$

in terms of a single trigonometric function, find $\sum_{x=1}^{n}\cos\left(x-\frac{1}{4}\right)\pi$, leaving your answer in terms of n. [4]

(b) The function f is defined by

$$\mathrm{f}: x \mapsto \sin\left(x+\frac{1}{4}\right)\pi-\sin\left(x-\frac{3}{4}\right)\pi,\ x\in\mathbb{R},\ \ a\le x\le 1.$$

(i) State the range of f and sketch the curve when $a=-1$, labelling the exact coordinates of the points where the curve crosses the x- and y- axes. [3]

(ii) State the least value of a such that f^{-1} exists, and define f^{-1} in similar form. [3]

The function g is defined by

$$\mathrm{g}: x \mapsto \frac{2x}{1-x},\ x\in\mathbb{R},\ \ x\ge\frac{13}{5}.$$

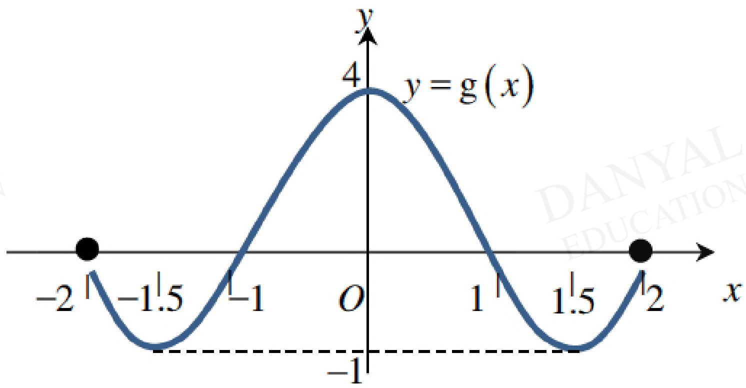
Given that fg exists, find the greatest value of a, and the corresponding range of fg. [3]

Q2

The function f is defined by

$$f : x \mapsto (x-k)^2 \ , \ x < k \ \text{ where } \ k > 5 .$$

(i) Find $f^{-1}(x)$ and state the domain of f^{-1}. [3]



The diagram above shows the curve with equation $y = g(x)$, where $-2 \le x \le 2$. The curve crosses the x–axis at $x = -2$, $x = -1$, $x = 1$ and $x = 2$, and has turning points at $(-1.5, -1)$, $(0, 4)$ and $(1.5, -1)$.

(ii) Explain why the composite function fg exists. [2]

(iii) Find in terms of k,

(a)the value of fg (-1) [1]

(b)the range of fg. [2]

Q3

The functions f and g are defined by

$$\mathrm{f}: x \mapsto 2x^2 - x, \quad x \in \mathbb{R}, \quad x \geq 0,$$

$$\mathrm{g}: x \mapsto -3 + \frac{1}{\sqrt{2x + \frac{1}{2}}}, \quad x \in \mathbb{R}, \quad x > -\frac{1}{4}.$$

(i) Give a reason why f does not have an inverse. [1]

(ii) If the domain of f is restricted to $x \geq k$, state the least value of k for which the function f^{-1} exists, and find f^{-1} in similar form for this domain. [3]

(iii) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram if the domain of f is restricted to $x \geq k$, where k is the value found in part **(ii)**. Your diagram should show clearly the relationship between the two graphs. [3]

(iv) Solve algebraically the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for the restricted domain of f in part **(ii)**. [2]

(v) For f defined for $x \geq 0$, show that the composite function gf exists and find its range. [3]

Answers

Functions Test 1

Q1

(a)

By factor formula,

$$\sin\left(x+\tfrac{1}{4}\right)\pi-\sin\left(x-\tfrac{3}{4}\right)\pi=2\cos\left[\tfrac{1}{2}\left(2x-\tfrac{1}{2}\right)\pi\right]\sin\left(\tfrac{1}{2}\pi\right)$$
$$=2\cos\left(x-\tfrac{1}{4}\right)\pi.$$

Many students expanded each term using compound angle formula then tried to collapse the terms back into one trig function, mostly without success.

The **most common error** was to first factorise π out of the expression then use factor formula:

$$\sin\left(x+\tfrac{1}{4}\right)\pi-\sin\left(x-\tfrac{3}{4}\right)\pi$$
$$=\pi\left[\sin\left(x+\tfrac{1}{4}\right)-\sin\left(x-\tfrac{3}{4}\right)\right]$$

which is ridiculous.

Students need to realise that this is a 1-mark question which should not require page-long working.

Hence

$$\sum_{x=1}^{n}2\cos\left(x-\tfrac{1}{4}\right)\pi$$
$$=\sum_{x=1}^{n}\left[\sin\left(x+\tfrac{1}{4}\right)\pi-\sin\left(x-\tfrac{3}{4}\right)\pi\right]$$
$$=\left[\sin\tfrac{5}{4}\pi-\sin\tfrac{1}{4}\pi\right]+\left[\sin\tfrac{9}{4}\pi-\sin\tfrac{5}{4}\pi\right]+\dots$$
$$+\left[\sin\left(n-\tfrac{3}{4}\right)\pi-\sin\left(n-\tfrac{7}{4}\right)\pi\right]+\left[\sin\left(n+\tfrac{1}{4}\right)\pi-\sin\left(n-\tfrac{3}{4}\right)\pi\right]$$
$$=\sin\left(n+\tfrac{1}{4}\right)\pi-\sin\tfrac{1}{4}\pi$$
$$=\sin\left(n+\tfrac{1}{4}\right)\pi-\tfrac{1}{\sqrt{2}}$$

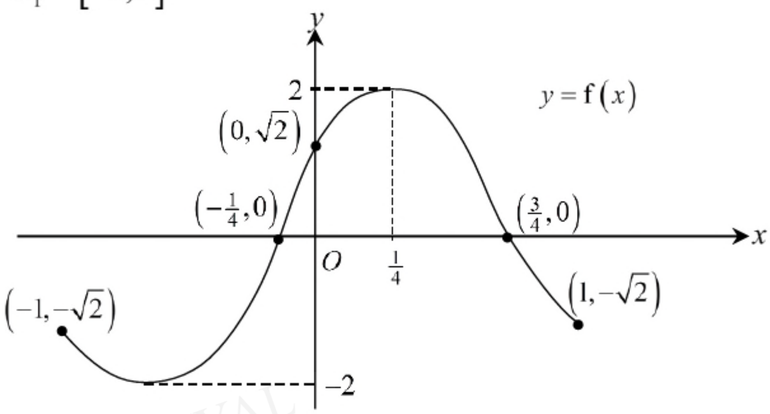
Therefore,

$$\sum_{x=1}^{n}\cos\left(x-\tfrac{1}{4}\right)\pi=\tfrac{1}{2}\sin\left(n+\tfrac{1}{4}\right)\pi-\tfrac{1}{2\sqrt{2}}.$$

Those who couldn't do the first part naturally were not able to do this part accurately.

Amongst those who did, some evaluated the value of each trigo expression and hence could not see which terms cancelled out using the method of difference:

$$\sum_{x=1}^{n}\left[\sin\left(x+\tfrac{1}{4}\right)\pi-\sin\left(x-\tfrac{3}{4}\right)\pi\right]$$
$$=\left[-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right]+\left[\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right]+\dots$$
$$+\left[\sin\left(n-\tfrac{3}{4}\right)\pi-\sin\left(n-\tfrac{7}{4}\right)\pi\right]+\left[\sin\left(n+\tfrac{1}{4}\right)\pi-\sin\left(n-\tfrac{3}{4}\right)\pi\right]$$

(b)(i)	$R_{\mathrm{f}}=[-2,2]$ 	Biggest problem for the plot is students keying in to G.C. wrongly. Plotting $Y=\sin\left(X+\frac{1}{4}\right)\pi-\sin\left(X-\frac{3}{4}\right)\pi$ instead of $Y=\sin\left[\left(X+\frac{1}{4}\right)\pi\right]-\sin\left[\left(X-\frac{3}{4}\right)\pi\right]$ Students should be careful, using brackets when appropriate. Once the graph is correctly plotted in the G.C. with the correct domain, they should notice that one full period is plotted, and that the range is easily read off the G.C.
(b)(ii)	Least value of a is $\frac{1}{4}$. Let $y=2\cos\left(x-\frac{1}{4}\right)\pi$. Then $\cos^{-1}\left(\frac{y}{2}\right)=\left(x-\frac{1}{4}\right)\pi \quad \Rightarrow \quad x=\dfrac{\cos^{-1}\left(\frac{y}{2}\right)}{\pi}+\dfrac{1}{4}$. $\therefore\ \mathrm{f}^{-1}: x \mapsto \dfrac{1}{\pi}\cos^{-1}\left(\dfrac{x}{2}\right)+\dfrac{1}{4}, \qquad x\in\left[-\sqrt{2},2\right]$	If graph is correctly sketched, least value of a is easily found. Method mark for making x the subject of $y=2\cos\left(x-\frac{1}{4}\right)\pi$ is awarded for any attempt to find the inverse function, regardless of whether students' graphs are sketched correctly. Many students were careless in either not quoting the domain of f^{-1} or, for those who did, quoted it forgetting that domain of f is now restricted so that its inverse exsits.
(b)(iii)	fg exists $\quad \Rightarrow \quad R_{\mathrm{g}}\subseteq D_{\mathrm{f}}$ now $R_{\mathrm{g}}=\left[-\dfrac{13}{4},-2\right)$ and $D_{\mathrm{f}}=[a,1]$ since fg exists, $a\le-\dfrac{13}{4}$. Hence the greatest value of a is $-\frac{13}{4}$. $R_{\mathrm{fg}}=\mathrm{f}\left(R_{\mathrm{g}}\right)=\mathrm{f}\left[-\dfrac{13}{4},-2\right)=\left[-2,\sqrt{2}\right]$.	Students were not tenacious enough to find R_{g} properly, perhaps discouraged from the earlier parts. g is a straight forward function that can be sketched with the G.C., bearing in mind that there is a horizontal asymptote at $y=-2$.

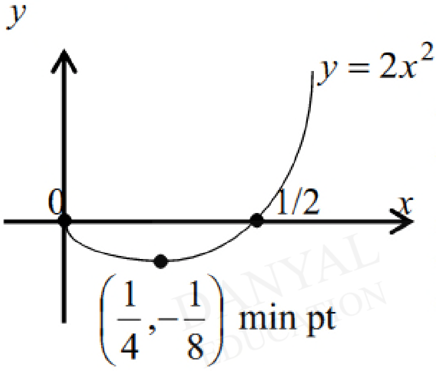
Q2

(i)	Let $y=(x-k)^2$ $x-k=\pm\sqrt{y}$ $x=-\sqrt{y}+k \quad (\because x<k)$ $f^{-1}(x)=-\sqrt{x}+k$ $D_{f^{-1}}=(0,\ \infty)$
(ii)	$R_g=[-1,\ 4]$ $D_f=(-\infty,\ k)$ Since $k>5$, $R_g \subseteq D_f$. Thus fg exists.
(iii)	$fg(-1)=f(0)=k^2$ Using $R_g=[-1,\ 4]$, and the fact that f is a strictly decreasing function in the given domain, $R_{fg}=\left[(4-k)^2,\ (-1-k)^2\right]$ $=\left[(4-k)^2,\ (1+k)^2\right]$

Q3

(i)

As shown in the following sketch:



Any horizontal line of the form $y = k$ where $-\frac{1}{4} < k \leq 0$ will intersect the curve at 2 points.

Thus, f is not one-one and hence f^{-1} does not exist.

(ii)

From the sketch of the curve, we deduce that the least value of $k = \frac{1}{4}$ for f^{-1} to exist.

Next let $y = 2x^2 - x$. Then we have

$$y = 2\left(x^2 - \frac{1}{2}x\right)$$

$$= 2\left(x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right)$$

$$= 2\left(x - \frac{1}{4}\right)^2 - \frac{1}{8}$$

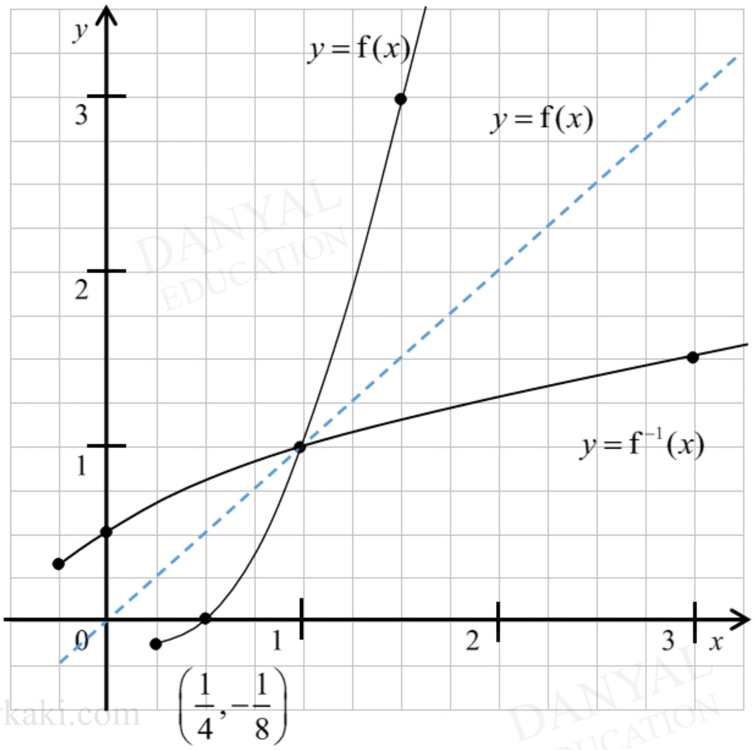
$$\left(x - \frac{1}{4}\right)^2 = \frac{1}{2}\left(y + \frac{1}{8}\right)$$

$$\Rightarrow x = \frac{1}{4} \pm \sqrt{\frac{8y+1}{16}} = \frac{1}{4} + \frac{\sqrt{8y+1}}{4} \qquad \text{since } x \geq \frac{1}{4}$$

Hence, $f^{-1}: x \mapsto \dfrac{1+\sqrt{8x+1}}{4}, x \geq -\dfrac{1}{8}$. $D_{f^{-1}} = R_f = \left[-\dfrac{1}{8}, \infty\right)$

(iii)

Sketch of $y = f(x)$ and $y = f^{-1}(x)$:



(iv)

From the sketch in part (iii) we note that to solve the equation $f(x) = f^{-1}(x)$, we can also solve $f(x) = x$

Thus, $2x^2 - x = x \Rightarrow 2x(x-1) = 0$

Therefore, in the restricted domain of $x \geq \dfrac{1}{4}$,

the solution is $x = 1$.

(v)

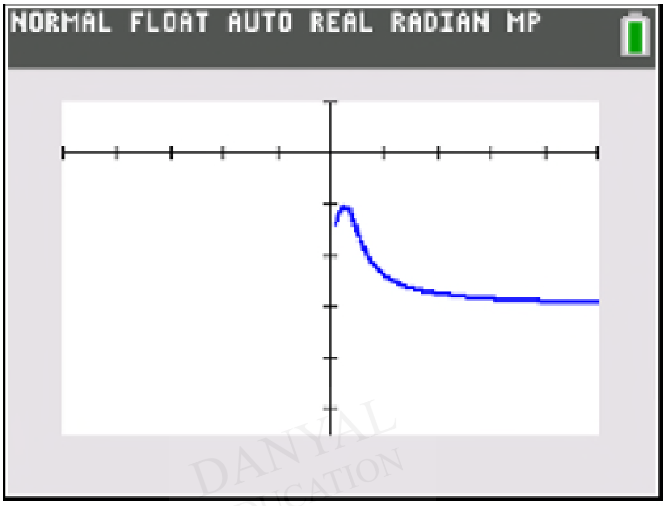
For $f: x \mapsto 2x^2 - x,\ x \in \mathbb{R},\ x \geq 0, R_f = \left[-\dfrac{1}{8}, \infty\right)$

Also, for $g: x \mapsto -3 + \dfrac{1}{\sqrt{2x+\dfrac{1}{2}}},\ x \in \mathbb{R},\ x > -\dfrac{1}{4},\ D_g = \left(-\dfrac{1}{4}, \infty\right)$

Since $R_f \subseteq D_g$, the composite function gf exists.

Then,

$$gf(x) = -3 + \frac{1}{\sqrt{2(2x^2-x)+\frac{1}{2}}} = -3 + \frac{1}{\sqrt{4\left(x-\frac{1}{4}\right)^2+\frac{1}{4}}}$$



Since $D_{gf} = [0, \infty)$ and $gf\left(\frac{1}{4}\right) = -3 + \frac{1}{\sqrt{\frac{1}{4}}} = -3 + 2 = -1$,

we have $R_{gf} = (-3, -1]$

<u>ALT</u>

$$[0, +\infty) \to [-\frac{1}{8}, +\infty) \to (-3, -1]$$

$$R_{gf} = (-3, -1]$$