A Level H2 Math

Discrete Random Variable Test 5

Q1

Four digits are randomly selected from the set $\{1,2,3,4,5,6,7,8,9\}$ to form a four-digit number. Repetitions are not allowed.

(i) Find the probability that none of the digits in the four-digit number are odd. [2]

The random variable X denotes the number of odd digits in the four-digit number formed.

(ii) Show that $\mathrm{P}(X=1)=\frac{10}{63}$, and find the rest of the probability distribution of X, giving each probability as a fraction in its lowest terms. [3]

(iii) Find the expectation and variance of X. [3]

(iv) Two independent observations of X are denoted by X_1 and X_2.
Find $\mathrm{P}(|X_1-X_2|<3)$. [4]

Q2

From past records, the number of days of hospitalization for an individual with minor ailment can be modelled by a discrete random variable with probability density function given by

$$P(X=x)=\begin{cases}\frac{6-x}{15}, & \text{for } x=1,2,3,4,5,\\ 0, & \text{otherwise.}\end{cases}$$

An insurance policy pays \$100 per day for up to 3 days of hospitalization and \$25 per day of hospitalization thereafter.

(i) Calculate the expected payment for hospitalization for an individual under this policy. [4]

(ii) The insurance company will incur a loss if the total payout for 100 hospitalisation claims under this policy exceed \$24000. Using a suitable approximation, estimate the probability that the insurance company will incur a loss for 100 such claims. [4]

Q3

The probability function of X is given by

$$P(X = x) = \begin{cases} (2x-1)\theta & \text{if } x = 1, 2, 3 \\ k & \text{if } x = 4 \\ 0 & \text{otherwise} \end{cases}$$

where $0 < \theta < \dfrac{1}{9}$.

(i) Show that $k = 1 - 9\theta$. Find, in terms of θ, the probability distribution of X. [2]

(ii) Find $E(X)$ in terms of θ and hence show that $\text{Var}(X) = 26\theta - 196\theta^2$. [3]

(iii) The random variable Y is related to X by the formula $Y = a + bX$, where a and b are non-zero constants. Given that $\text{Var}(Y) = \dfrac{1}{3}b^2$, find the value of θ. [3]

Answers
Discrete Random Variable Test 5

Q1

i P(no odd digits) = P(all even digits)

$$= \frac{{}^4C_4}{{}^9C_4} \left(\text{or } \frac{{}^4P_4}{{}^9P_4} \right)$$

$$= \frac{1}{126}$$

ii

$$P(X=1) = \frac{{}^5C_1\,{}^4C_3}{{}^9C_4} = \frac{10}{63}$$

x	0	1	2	3	4
$P(X=x)$	$\frac{1}{126}$	$\frac{10}{63}$	$\frac{10}{21}$	$\frac{20}{63}$	$\frac{5}{126}$

iii

$$E(X) = \sum_{x=0}^{4} x P(X=x)$$

$$= 1\left(\frac{10}{63}\right) + 2\left(\frac{10}{21}\right) + 3\left(\frac{20}{63}\right) + 4\left(\frac{5}{126}\right)$$

$$= \frac{20}{9}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \sum_{x=0}^{4} x^2 P(X=x) - \left(\frac{20}{9}\right)^2$$

$$= 1\left(\frac{10}{63}\right) + 4\left(\frac{10}{21}\right) + 9\left(\frac{20}{63}\right) + 16\left(\frac{5}{126}\right) - \left(\frac{20}{9}\right)^2$$

$$= \frac{50}{81}$$

iv

$$\begin{aligned}
P(|X_1 - X_2| < 3) &= P(-3 < X_1 - X_2 < 3) \\
&= 1 - 2P(X_1 = 0 \quad \& \quad X_2 = 4) \\
&\quad -2P(X_1 = 0 \quad \& \quad X_2 = 3) \\
&\quad -2P(X_1 = 1 \quad \& \quad X_2 = 4) \\
&= 1 - 2\left(\frac{1}{126}\right)\left(\frac{5}{126}\right) - 2\left(\frac{1}{126}\right)\left(\frac{20}{63}\right) \\
&\quad -2\left(\frac{10}{63}\right)\left(\frac{5}{126}\right) \\
&= \frac{7793}{7938} \quad (\text{or } 0.982)
\end{aligned}$$

Q2

(i) Let Y be the payment for an individual. The probability table is as follows:

y	100	200	300	325	350
$P(Y=y)$	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{15}$

$$\mathrm{E}(Y) = 100\cdot\frac{1}{3} + 200\cdot\frac{4}{15} + 300\cdot\frac{1}{5} + 325\cdot\frac{2}{15} + 350\cdot\frac{1}{15}$$
$$= 213\frac{1}{3}$$

Candidates must read the question carefully to understand the payment scale, and thereafter to write down the probability table for Y correctly. Note that there is no linear relationship between Y and X. Thus working out $E(X)$ will not obtain any credit.

(ii)
$$\mathrm{E}(Y^2) = 100^2\cdot\frac{1}{3} + 200^2\cdot\frac{4}{15} + 300^2\cdot\frac{1}{5} + 325^2\cdot\frac{2}{15} + 350^2\cdot\frac{1}{15}$$
$$= 54250$$

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - [\mathrm{E}(Y)]^2 = 8738\frac{8}{9}$$

Since $n = 100$ is large, by Central Limit Theorem,

$$T = \sum_{i=1}^{100} Y_i \sim \mathrm{N}(21333.33, 873888.89) \text{ approx.}$$

$$\text{Prob. Req'd} = P(T > 24000)$$
$$\approx 0.00217$$

Note that $E(Y^2)$ is not the variance of Y. Further, candidates are reminded that you should not assume that Y is normally distributed. Central Limit Theorem is necessary to obtain the approximate distribution of ΣY_i.

Q3

(i)

x	1	2	3	4
$\mathrm{P}(X=x)$	θ	3θ	5θ	k

Since $\sum_{\text{all } x} \mathrm{P}(X=x)=1$,

$\theta+3\theta+5\theta+k=1$

$\therefore k=1-9\theta$

Probability distribution of X is

x	1	2	3	4
$\mathrm{P}(X=x)$	θ	3θ	5θ	$1-9\theta$

(ii)

$\mathrm{E}(X)=1(\theta)+2(3\theta)+3(5\theta)+4(1-9\theta)$

$=\theta+6\theta+15\theta+4-36\theta$

$=4-14\theta$

$\mathrm{E}(X^2)=1^2(\theta)+2^2(3\theta)+3^2(5\theta)+4^2(1-9\theta)$

$=\theta+12\theta+45\theta+16-144\theta$

$=16-86\theta$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$

$=16-86\theta-(4-14\theta)^2$

$=16-86\theta-(16-112\theta+196\theta^2)$

$=26\theta-196\theta^2$

(iii)

$Y=a+bX$

$\mathrm{Var}(Y)=\mathrm{Var}(a+bX)$

$\mathrm{Var}(Y)=b^2\mathrm{Var}(X)$

$\frac{1}{3}b^2=b^2(26\theta-196\theta^2)$

$196\theta^2-26\theta+\frac{1}{3}=0 \quad (\because b\neq 0)$

Using GC,

$\theta=0.0144$ or $\theta=0.118$ (rejected $\because 0<\theta<\frac{1}{9}$)