A Level H2 Math

Discrete Random Variable Test 3

Q1

A biased tetrahedral (4-sided) die has its faces numbered '-1', '0', '2' and '3'. It is thrown onto a table and the random variable X denotes the number on the face in contact with the table. The probability distribution of X is as shown.

x	-1	0	2	3
$\mathrm{P}(X=x)$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$

(i) The random variable Y is defined by X_1+X_2, where X_1 and X_2 are 2 independent observations of X. Show that $\mathrm{P}(Y=2)=\frac{3}{16}$. [2]

(ii) In a game, a player pays \$2 to throw two such biased tetrahedral dice simultaneously on a table. For each die, the number on the face in contact with the table is the score of the die. The player receives \$16 if the maximum of the two scores is -1, and receives \$3 if the sum of the two scores is prime. For all other cases, the player receives nothing. Find the player's expected gain in the game. [4]

Q2

The probability distribution of a discrete random variable, X, is shown below.

x	1	2
$\mathrm{P}(X=x)$	a	b

Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ in terms of a. [5]

Q3

A random variable X has the probability distribution given in the following table.

x	2	3	4	5
$\mathrm{P}(X=x)$	0.2	a	b	0.45

Given that $E(|X-4|)=\frac{11}{10}$, find the values of a and b. [3]

Two independent observations of X are taken. Find the probability that one of them is 2 and the other is at most 4. [2]

Answers
Discrete Random Variable Test 3

Q1

(i)

$$\mathrm{P}(Y=2)$$
$$=2\mathrm{P}(X_1=2 \text{ and } X_2=0)+2\mathrm{P}(X_1=3 \text{ and } X_2=-1)$$
$$=2\mathrm{P}(X_1=2)\mathrm{P}(X_2=0)+2\mathrm{P}(X_1=3)\mathrm{P}(X_2=-1)$$
$$=2\left(\frac{1}{8}\right)\left(\frac{1}{2}\right)+2\left(\frac{1}{8}\right)\left(\frac{1}{4}\right)$$
$$=\frac{3}{16}$$

(ii)

$$\mathrm{P}(\text{max of 2 scores}=-1)$$
$$=\mathrm{P}(X_1=-1)\mathrm{P}(X_2=-1)$$
$$=\left(\frac{1}{8}\right)^2$$
$$=\frac{1}{64}$$

When sum of scores is prime, then $Y=2$, 3 or 5 .

From (i), $\mathrm{P}(Y=2)=\dfrac{3}{16}$

$$\mathrm{P}(Y=3)=2\mathrm{P}(X_1=0)\mathrm{P}(X_2=3)$$
$$=2\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$$
$$=\frac{1}{4}$$

$$\mathrm{P}(Y=5)=2\mathrm{P}(X_1=3)\mathrm{P}(X_2=2)$$
$$=2\left(\frac{1}{8}\right)\left(\frac{1}{4}\right)$$
$$=\frac{1}{16}$$

$\therefore$ Expected gain

$$=16\left(\frac{1}{64}\right)+3\left(\frac{3}{16}+\frac{1}{4}+\frac{1}{16}\right)-2$$
$$=-0.25$$

Hence expected gain is $-\$0.25$.
[Or expected loss is $\$0.25$.]

Alternatively,
$\therefore$ Expected gain

$$=(16-2)\left(\frac{1}{64}\right)+(3-2)\left(\frac{3}{16}+\frac{1}{4}+\frac{1}{16}\right)-2\left[1-\left(\frac{1}{64}+\frac{3}{16}+\frac{1}{4}+\frac{1}{16}\right)\right]$$
$$=-0.25$$

Hence expected gain is $-\$0.25$.
[Or expected loss is $\$0.25$.]

Q2

$b = 1 - a$

x	1	2
$\mathrm{P}(X = x)$	a	$1-a$

$$\begin{aligned} \mathrm{E}(X) &= 1(a) + 2(1-a) \\ &= \underline{\underline{2-a}} \\ \mathrm{E}(X^2) &= 1^2(a) + 2^2(1-a) \\ &= 4 - 3a \\ \mathrm{Var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\ &= 4 - 3a - (2-a)^2 \\ &= 4 - 3a - (4 - 4a + a^2) \\ &= \underline{\underline{a - a^2}} \end{aligned}$$

Q3

$$\sum_{\text{all } x} \mathrm{P}(X = x) = 1 = 0.2 + a + b + 0.45 \Rightarrow a + b = 0.35 \quad \ldots (1)$$

$$E(|X-4|) = 1\frac{1}{10} \Rightarrow \sum_{\text{all } x} |x-4| \mathrm{P}(X = x) = \frac{11}{10}$$

$$\Rightarrow 2(0.2) + a + 0 + 0.45 = \frac{11}{10}$$

$$\Rightarrow a = 0.25 \text{ and } b = 0.1$$

$$\begin{aligned} \text{P(required)} &= \mathrm{P}(X_1 = 2, X_2 = 2) + 2[\mathrm{P}(X_1 = 2, X_2 = 3) + \mathrm{P}(X_1 = 2, X_2 = 4)] \\ &= 0.2 \times 0.2 + 2[0.2 \times 0.25 + 0.2 \times 0.1] \\ &= 0.18 \end{aligned}$$