A Level H2 Math

Discrete Random Variable Test 2

Q1

A box contains 2 red balls, 3 green balls and x blue balls, where $x \in \mathbb{Z}, x \geq 5$. A game is played where the contestant picks 5 balls from the box without replacement. The total score, S, for the contestant is the sum of the number of green balls chosen and thrice the number of red balls chosen. The blue balls will not contribute any points, unless all 5 balls are blue. If all the 5 balls are blue, the score will be 25 points.

(i) Show that $\mathrm{P}(S=6)=\dfrac{20x(x^2-3x+14)}{(x+5)(x+4)(x+3)(x+2)(x+1)}$. [2]

(ii) Given that $\mathrm{P}(S=6)=\dfrac{5}{63}$, calculate x. [2]

(iii) Complete the probability distribution table for S. [4]

s	1	2	3	4	5	6	7	8	9	25
$\mathrm{P}(S=s)$		$\frac{5}{42}$	$\frac{5}{63}$		$\frac{5}{21}$	$\frac{5}{63}$		$\frac{5}{84}$	$\frac{1}{252}$	

(iv) Evaluate $\mathrm{E}(S)$ and find the probability that S is more than $\mathrm{E}(S)$. [2]

(v) Find the probability that there are no green balls drawn given that S is more than $\mathrm{E}(S)$. [2]

Q2

There are three identically shaped balls, numbered from 1 to 3, in a bag. Balls are drawn one by one at random and with replacement. The random variable X is the number of draws needed for any ball to be drawn a second time. The two draws of the same ball do not need to be consecutive.

(i) Show that $\mathrm{P}(X = 4) = \dfrac{2}{9}$ and find the probability distribution of X. [3]

(ii) Show that $\mathrm{E}(X) = \dfrac{26}{9}$ and find the exact value of $\mathrm{Var}(X)$. [3]

(iii) The mean for forty-four independent observations of X is denoted by $\overline{X}$. Using a suitable approximation, find the probability that $\overline{X}$ exceeds 3. [3]

Q3

A new game has been designed for a particular casino using two fair die. In each round of the game, a player places a bet of \$2 before proceeding to roll the two die. The player's score is the sum of the results from both die. For the scores in the following table, the player keeps his bet and receives a payout as indicated.

Score	Payout
9 or 10	\$1
2 or 4	\$5
11	\$8

For any other scores, the player loses his bet.

Let X be the random variable denoting the winnings of the casino from each round of the game.

(i) Show that $\mathrm{E}(X)=\dfrac{1}{12}$ and find $\mathrm{Var}(X)$. [4]

(ii) $\bar{X}$ is the mean winnings of the casino from n rounds of this game. Find $\mathrm{P}(\bar{X}>0)$ when $n=30$ and $n=50\,000$. Make a comparison of these probabilities and comment in context of the question. [3]

Answers

Discrete Random Variable Test 2

Q1

(i)

$$\mathrm{P}(S=6)$$
$$=\mathrm{P}(RRBBB)+\mathrm{P}(RGGGB)$$
$$=\frac{{}^5C_3(2!)\left[x(x-1)(x-2)\right]}{(x+5)(x+4)(x+3)(x+2)(x+1)}+\frac{{}^5C_3x(2)(2)(3!)}{(x+5)(x+4)(x+3)(x+2)(x+1)}$$
$$=\frac{20x\left[x^2-3x+2\right]}{(x+5)(x+4)(x+3)(x+2)(x+1)}+\frac{20x(12)}{(x+5)(x+4)(x+3)(x+2)(x+1)}$$
$$=\frac{20x(x^2-3x+14)}{(x+5)(x+4)(x+3)(x+2)(x+1)}$$

This part is usually well done. Lack of essential working is not acceptable.

(ii)

$$\frac{5}{63}=\frac{20x(x^2-3x+14)}{(x+5)(x+4)(x+3)(x+2)(x+1)}$$
$$5(x+5)(x+4)(x+3)(x+2)(x+1)-1260x(x^2-3x+14)=0$$

Solving, the only integer root is $x=5$

This part is well done.

(iii)

s	1	4	7	25
$\mathrm{P}(S=s)$	$\frac{5}{84}$ or $\frac{15}{252}$	$\frac{5}{21}$ or $\frac{60}{252}$	$\frac{5}{42}$ or $\frac{30}{252}$	$\frac{1}{252}$

$$\mathrm{P}(S=1)=\mathrm{P}(\mathrm{GBBBB})$$
$$=\left(\frac{3}{10}\right)\left(\frac{5}{9}\right)\left(\frac{4}{8}\right)\left(\frac{3}{7}\right)\left(\frac{2}{6}\right)\left(\frac{5!}{4!}\right)$$
$$=\frac{5}{84}\text{ or }\frac{15}{252}$$

Note: $\frac{5!}{4!}$ is for arranging GBBBB with 4 repeated "B"s.

$\mathrm{P}(S=4)$ and $\mathrm{P}(S=7)$ proved to be quite challenging for most candidates.

$\mathrm{P}(S=4)=\mathrm{P}(\mathrm{RGBBB})$

$$=\left(\frac{2}{10}\right)\left(\frac{3}{9}\right)\left(\frac{5}{8}\right)\left(\frac{4}{7}\right)\left(\frac{3}{6}\right)\left(\frac{5!}{3!}\right)$$

$$=\frac{5}{21}\text{ or }\frac{60}{252}$$

Note: $\frac{5!}{3!}$ is for arranging RGBBB with 3 repeated "B"s.

$\mathrm{P}(S=7)=\mathrm{P}(\mathrm{RRGBB})$

$$=\left(\frac{2}{10}\right)\left(\frac{1}{9}\right)\left(\frac{3}{8}\right)\left(\frac{5}{7}\right)\left(\frac{4}{6}\right)\left(\frac{5!}{2!2!}\right)$$

$$=\frac{5}{42}\text{ or }\frac{30}{252}$$

Note: $\frac{5!}{2!2!}$ is for arranging RRGBB with 2 repeated "R"s and 2 repeated "B"s.

$\mathrm{P}(S=25)=\mathrm{P}(\mathrm{BBBBB})$

$$=\left(\frac{5}{10}\right)\left(\frac{4}{9}\right)\left(\frac{3}{8}\right)\left(\frac{2}{7}\right)\left(\frac{1}{6}\right)$$

$$=\frac{1}{252}$$

(iv) $\mathrm{E}(S)=4.60$ or $\frac{1159}{252}$

$\mathrm{P}(S>4.60)=\frac{127}{252}$ or 0.504

This part is poorly done as a result of errors from (iii).

(v) $\mathrm{P}(\text{no }G\mid S>4.60)$

$$=\frac{\mathrm{P}(RRBBB)+\mathrm{P}(BBBBB)}{\frac{127}{252}}$$

$$=\frac{\frac{20(5)(4)(3)}{(10)(9)(8)(7)(6)}+\frac{1}{252}}{\frac{127}{252}}$$

$$=\frac{\frac{10}{252}+\frac{1}{252}}{\frac{127}{252}}$$

$$=\frac{11}{127}$$

Common omission or error pertaining to the case $\mathrm{P}(RRBBB)$, led to wrong answer.
This part is poorly done.

Q2

(i)

$$\mathrm{P}(X=4)$$
$$=\frac{3}{3}\times\frac{2}{3}\times\frac{1}{3}\times\frac{3}{3}$$
$$=\frac{2}{9}$$

x	2	3	4
$\mathrm{P}(X=x)$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{2}{9}$

(ii)

$$\mathrm{E}(X)$$
$$=\frac{1}{3}\times2+\frac{4}{9}\times3+\frac{2}{9}\times4$$
$$=\frac{26}{9}$$

$$\mathrm{E}(X^2)$$
$$=\frac{1}{3}\times2^2+\frac{4}{9}\times3^2+\frac{2}{9}\times4^2$$
$$=\frac{80}{9}$$

Var(X)

$$=\frac{80}{9}-\left(\frac{26}{9}\right)^2$$
$$=\frac{44}{81}$$

(iii)

Since n =44 <u>is large</u>, by Central Limit Theorem, $\overline{X}\sim N\left(\frac{26}{9},\frac{44}{81}\div44\right)$ approx.

$$P\left(\overline{X}>3\right)$$
$$=0.159\ (\text{By GC})$$

Q3

(i)

x	- 8	- 5	- 1	2
$\text{P}(X=x)$	$\frac{2}{36}=\frac{1}{18}$	$\frac{4}{36}=\frac{2}{18}$	$\frac{7}{36}$	$\frac{23}{36}$

$$\text{E}(X)=\frac{46}{36}-\frac{7}{36}-\frac{20}{36}-\frac{16}{36}=\frac{1}{12}$$

$$\begin{aligned}\text{Var}(X)=\text{E}(X^2)-\left(\text{E}(X)\right)^2&=\frac{92}{36}+\frac{7}{36}+\frac{50}{18}+\frac{64}{18}-\frac{1}{12^2}\\&=\frac{1307}{144}\text{ or }9.08\text{ (to 3sf)}\end{aligned}$$

(ii) Since n is large, $\overline{X}\sim\text{N}\left(\frac{1}{12},\frac{1307}{144n}\right)$ approximately by Central Limit Theorem.

For $n=30$, $\text{P}\left(\overline{X}>0\right)=0.560$ (to 3sf)

For $n=50000$, $\text{P}\left(\overline{X}>0\right)=1.00$ (to 3sf)

The more rounds this game is played, the higher the chance of casino receiving a positive average winnings. In other words, it is almost certain that casino will win in the long run.