A Level H2 Math

Discrete Random Variable Test 1

Q1

Alex and his friend stand randomly in a queue with 3 other people. The random variable X is the number of people standing between Alex and his friend.

(i) Show that $P(X = 2) = 0.2$. [2]

(ii) Tabulate the probability distribution of X. [2]

(iii) Find $E(X)$ and $E(X - 1)^2$. Hence find Var (X). [3]

Q2

An unbiased six-sided die is rolled twice. The random variable X represents the higher of the two values if they are different, and their common value if they are the same. The probability distribution of X is given by the formula

$$P(X = r) = k(2r - 1) \quad \text{for } r = 1, 2, 3, 4, 5, 6.$$

(i) Find the exact value of k, giving your answer as a fraction in its simplest form. [2]

(ii) Find the expectation of X. [2]

A round of the game consists of rolling the unbiased six-sided die twice, and X is taken as the score for the round. A player plays three rounds of the game.

(iii) Find the probability that the total score for the three rounds is 16. [2]

Q3

A board game simulates players attacking each other by throwing tetrahedral (8-sided) dice. When attacking, the player throws an attack die once. An attack die has 5 of the sides printed with the number "0", 2 of the sides printed with the number "1", and 1 of the sides printed with the number "2". After the attacking player has thrown the attack die, the defending player throws a defence die once. A defence die has 2 of the sides printed with the number "0", 4 of the sides printed with the number "1" and 2 of the sides printed with the number "2". The damage dealt during a round is equal to the score shown on the attack die minus the score shown on the defence die. If the score on the defence die is more than the score on the attack die, the damage dealt will be zero.

Let A denotes the score on an attack die, and D denotes the score on a defence die.

(i) Write down the probability distributions for A and D. Hence find the expected value and variance of $A - D$. [4]

Let X denote the damage dealt during a round.

(ii) Find the probability distribution for X. Hence find the expected value and variance of X. [5]

(iii) Explain why, in the context of the question, $\text{E}(X) > 0$ when $\text{E}(A) < \text{E}(D)$. [1]

Answers

Discrete Random Variable Test 1

Q1

(i)

$$P(X=2)=P(A**F*,*A**F)=2\left(\frac{2\times 3!}{5!}\right)=\frac{1}{5}=0.2 \quad \text{(shown)}$$

(ii)

$$P(X=0)=P(AF***,*AF**,**AF*,***AF)=4\left(\frac{2\times 3!}{5!}\right)=\frac{2}{5}$$
$$=0.4$$

$$P(X=1)=P(A*F**,*A*F*,**A*F)=3\left(\frac{2\times 3!}{5!}\right)=\frac{3}{10}=0.3$$

$$P(X=3)=P(A***F)=\left(\frac{2\times 3!}{5!}\right)=\frac{1}{10}=0.1$$

x	0	1	2	3
$P(X=x)$	0.4	0.3	0.2	0.1

6(i) and (ii) were very crucial parts to this question. Students who were unable to start finding the pdf of X, or did it wrongly, would not have been able to answer (iii).

Some students lost marks for (i) because they lacked sufficient elaboration, e.g. writing simply 4/20 or 2/10 without justifying how they arrived at these numbers. They would have gotten the mark if they had drawn some diagram of how there are 4 ways of arranging Alex and his friend 2 persons apart (ignoring the arrangement of the other 3 people).

A significant number of students assumed X was a binomial random variable.

Students are also reminded to present sufficient working for the other probabilities in the table.

(iii)

$$E(X)=\sum_{\text{all } x} xP(X=x)=0(0.4)+1(0.3)+2(0.2)+3(0.1)=1$$

$$E(X-1)^2=\sum_{\text{all } x}(x-1)^2P(X=x)=1(0.4)+0(0.3)+1(0.2)+4(0.1)=1$$

$$\text{Var}(X)=E(X-\mu)^2=E(X-1)^2=1$$

This part was generally well done. Most of the errors came from the formula for $E(X-1)^2$. A variety of methods were seen for calculating $\text{Var}(X)$, but very few students figured out the shortest method: by the definition of $\text{Var}(X)$, which is $E[(X-\mu)^2]$.

One way for students to check if their answer for $\text{Var}(X)$ is correct is to know that variance cannot be a negative number.

Q2

(i)	$\sum_{r=1}^{6} \mathrm{P}(X=r)=1$ $k+3k+5k+7k+9k+11k=1$ $k=\frac{1}{36}$
(ii)	$\mathrm{E}(X)=1(k)+2(3k)+3(5k)+4(7k)+5(9k)+6(11k)$ $=161k$ $=\frac{161}{36}$
(iii)	Required Probability $=\mathrm{P}(\{6,6,4\})+\mathrm{P}(\{6,5,5\})$ $=\left(\frac{11}{36}\right)^2\left(\frac{7}{36}\right)\frac{3!}{2!}+\left(\frac{11}{36}\right)\left(\frac{9}{36}\right)^2\frac{3!}{2!}$ $=0.112$ (3 s.f.) Accept: $\frac{1738}{15552}=\frac{869}{7776}$

Q3

(i)

Probability distribution for A:

a	0	1	2
$P(A = a)$	5/8	2/8	1/8

Probability distribution for D:

d	0	1	2
$P(D = d)$	2/8	4/8	2/8

$$E(A) = \left(\frac{5}{8}\right)(0) + \left(\frac{2}{8}\right)(1) + \left(\frac{1}{8}\right)(2) = \frac{1}{2}$$

$$E(D) = \left(\frac{2}{8}\right)(0) + \left(\frac{4}{8}\right)(1) + \left(\frac{2}{8}\right)(2) = 1$$

$$E(A - D) = E(A) - E(D) = \frac{-1}{2}$$

$$E(A^2) = \left(\frac{5}{8}\right)(0)^2 + \left(\frac{2}{8}\right)(1)^2 + \left(\frac{1}{8}\right)(2)^2 = \frac{3}{4}$$

$$E(D^2) = \left(\frac{2}{8}\right)(0)^2 + \left(\frac{4}{8}\right)(1)^2 + \left(\frac{2}{8}\right)(2)^2 = \frac{3}{2}$$

$$\text{Var}(A)=\text{E}(A^2)-\text{E}(A)^2=\frac{3}{4}-\left(\frac{1}{2}\right)^2=\frac{1}{2}$$

$$\text{Var}(D)=\text{E}(D^2)-\text{E}(D)^2=\frac{3}{2}-1^2=\frac{1}{2}$$

$$\text{Var}(A-D)=\text{Var}(A)+\text{Var}(D)=\frac{1}{2}+\frac{1}{2}=1$$

(ii)

Probability distribution for X:

x	0	1	2
$\text{P}(X=x)$	$\text{P}(A=0)$ $+\text{P}(A=1)\text{P}(D\geq 1)$ $+\text{P}(A=2)\text{P}(D=2)$ $=\frac{27}{32}$	$\text{P}(A=1)\text{P}(D=0)$ $+\text{P}(A=2)\text{P}(D=1)$ $=\frac{1}{8}$	$\text{P}(A=2)\text{P}(D=0)$ $=\frac{1}{32}$

$$\text{E}(X)=\left(\frac{27}{32}\right)(0)+\left(\frac{1}{8}\right)(1)+\left(\frac{1}{32}\right)(2)=\frac{3}{16}$$

$$\text{E}(X^2)=\left(\frac{27}{32}\right)(0)^2+\left(\frac{1}{8}\right)(1)^2+\left(\frac{1}{32}\right)(2)^2=\frac{1}{4}$$

$$\text{Var}(X)=\frac{1}{4}-\left(\frac{3}{16}\right)^2=\frac{55}{256}$$

(iii)
If the score on the defence die is more than the score on the attack die, the damage dealt will be zero. So even though sometimes $A-D$ will be less than zero, that is never considered when dealing damage. Hence, the expected damage must be greater than zero.