A Level H2 Math

Differentiation and its Applications Test 9

Q1

The curve C has equation $2x-y^2=(x+y)^2$.

(i) Find the equations of the tangents to C which are parallel to the x-axis. [4]

(ii) The line l is tangent to C at $A(2,-2)$. If the normal to C at the origin O meets l at the point B, find the area of triangle OAB. [4]

Q2

A car is travelling at a speed of 30 m/s on a road heading towards a perpendicular train track, which is elevated 30 m above the ground. The front of the car is 40 m away from the track when the front of the train first crossed the road.

If the train is travelling at 20 m/s, show that the distance between the front of the train and the car is $\sqrt{1300t^2-2400t+2500}$ m. [2]

(i) How fast is the front of both the train and the car separating 1 second later? [2]

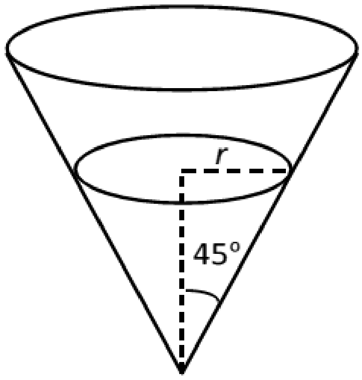
(ii) Find the distance when the front of the train and the front of the car are closest. [4]

(iii) Find the rate of change of the angle of elevation of the front of the train from the car 1 second later. [4]

Q3

Water is leaking at a rate of 2 cm^3 per minute from a container in the form of a cone, with its axis vertical and vertex downwards. The semi-vertical angle of the cone is 45° (see diagram). At time *t* minutes, the radius of the water surface is *r* cm. Find the rate of change of the depth of water when the depth of water in the container is 0.3cm. [4]

[The volume of a cone of base radius *r* and height *h* is given by $V = \frac{1}{3}\pi r^2 h$.]



Answers

Differentiation and its Applications Test 9

Q1

(i)

Differentiating $2x - y^2 = (x+y)^2$ ______(1)

implicitly with respect to x,

$$2 - 2y\frac{dy}{dx} = 2(x+y)\left(1+\frac{dy}{dx}\right)$$

Where tangent is parallel to the x-axis, $\frac{dy}{dx} = 0$.

$$2 = 2(x+y)$$

$$y = 1 - x \qquad \text{\_\_\_\_\_\_(2)}$$

Sub (2) in (1),

$$2x - (1-x)^2 = (x+1-x)^2$$

$$x^2 - 4x + 2 = 0$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)}}{2} = 2 \pm \sqrt{2}$$

When $x = 2 - \sqrt{2}$, $y = 1 - (2 - \sqrt{2}) = -1 + \sqrt{2}$

When $x = 2 + \sqrt{2}$, $y = 1 - (2 + \sqrt{2}) = -1 - \sqrt{2}$

(ii)

$$2 - 2y\frac{dy}{dx} = 2(x+y)\left(1+\frac{dy}{dx}\right)$$

$$2 = 2(x+y) + 2(x+2y)\frac{dy}{dx}$$

$$2 = 2(x+y) + 2(x+2y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1-(x+y)}{x+2y}$$

When $x = 0,\ y = 0$, $-\frac{1}{\frac{dy}{dx}} = 0$.

Hence normal to C at the origin is $y = 0$.

When $x = 2,\ y = -2$, $\frac{dy}{dx} = \frac{1}{-2}$

Tangent to C at $A(2,-2)$, $y - (-2) = -\frac{1}{2}(x-2)$

Where the normal and the tangent intersect,

$$2 = -\frac{1}{2}(x-2)$$

$$x = -2$$

Area of triangle $OAB = \frac{1}{2}(2)(2) = 2$ units^2

Q2

	Let s be the distance between the front of the car and the train at time t. $s=\sqrt{x^2+30^2}$ and $x^2=(40-30t)^2+(20t)^2$ $s=\sqrt{(40-30t)^2+(20t)^2+30^2}=\sqrt{1300t^2-2400t+2500}$
(i)	$\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600t-2400)$ When $t=1, \frac{\mathrm{d}s}{\mathrm{d}t}=2.67$
(ii)	At stationary point $\frac{\mathrm{d}s}{\mathrm{d}t}=0$ $\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600t-2400)=0$ $2600t-2400=0 \Rightarrow t=\frac{12}{13}$ $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{1}{2}}(2600)$ $+(-\frac{1}{2})\frac{1}{2}(1300t^2-2400t+2500)^{-\frac{3}{2}}(2600t-2400)$ When $t=\frac{12}{13}, \frac{d^2s}{dt^2}>0$ $s=\sqrt{1300(\frac{12}{13})^2-2400(\frac{12}{13})+2500}=$ 19.884=19.9
(iii)	Let the angle of elevation be θ $\sin\theta=\frac{30}{\sqrt{1300t^2-2400t+2500}}$ $\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{1}{2}(30)(1300t^2-2400t+2500)^{-\frac{3}{2}}(2600t-2400)$ When $t=1$, $\cos\theta=\frac{\sqrt{(40-30)^2+20^2}}{\sqrt{1300-2400+2500}}=\sqrt{\frac{500}{1400}}$ $\therefore \frac{\mathrm{d}\theta}{\mathrm{d}t}= -\frac{1}{2}(30)(1300-2400+2500)^{-\frac{3}{2}}(200)\div\sqrt{\frac{500}{1400}}$ =−0.0958 rad/s (or -5.5°/s)

Q3

$$\tan 45^\circ = \frac{r}{h} \Rightarrow r = h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi h^3$$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \pi h^2$$

When $h = 0.3$,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V}\frac{\mathrm{d}V}{\mathrm{dt}}$$

$$= \frac{1}{\pi(0.3)^2}(-2)$$

$$= -\frac{200}{9\pi} = -7.07 \ \ (3\text{s.f})$$

The depth of water is decreasing at 7.07 cm per minute.