A Level H2 Math

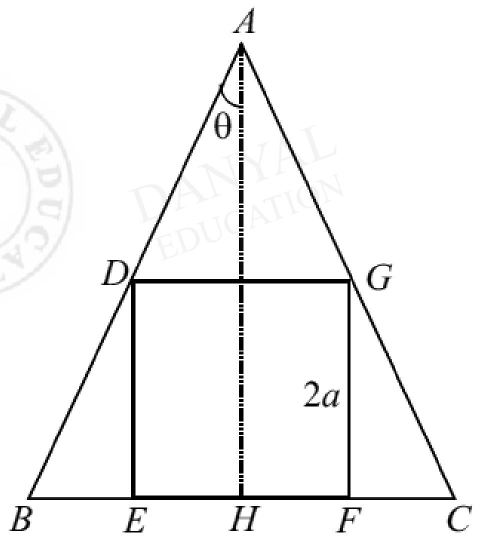
Differentiation and its Applications Test 8

Q1

The volume of a spherical bubble is increasing at a constant rate of λ cm^3 per second. Assuming that the initial volume of the bubble is negligible, find the exact rate in terms of λ at which the surface area of the bubble is increasing when the volume of the bubble is 20 cm^3. **[5]**

[The volume of a sphere, $V = \frac{4}{3}\pi r^3$ and the surface area of a sphere, $A = 4\pi r^2$ where r is the radius of the sphere.]

Q2



It is given that $DEFG$ is a square with fixed side $2a$ cm and it is inscribed in the isosceles triangle ABC with height AH, where $AB = AC$ and angle $BAH = \theta$.

(i) Taking $t = \tan\theta$, show that the area of the triangle ABC is given by $S = a^2\left(4 + 4t + \frac{1}{t}\right)$ [3]

(ii) Find the minimum area of S in terms of a when t varies. [4]

(iii) Hence sketch the graph showing the area of the triangle ABC as θ varies. [3]

Q3

A curve C is defined by the parametric equations

$$x = \frac{t}{1+t}, \qquad y = \frac{t^2}{1+t},$$

where t takes all real values except -1.

Find $\frac{dy}{dx}$, leaving your answer in terms of t. [3]

(i) Show that the equation of the tangent to C at the point $\left(\frac{p}{1+p}, \frac{p^2}{1+p}\right)$ is

$$y = p(p+2)x - p^2.$$ [2]

(ii) Find the acute angle between the two tangents to C which pass through the point $(2,5)$. [3]

Answers

Differentiation and its Applications Test 8

Q1

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

When V=20,

$$20 = \frac{4}{3}\pi r^3$$

$$r = \left(\frac{15}{\pi}\right)^{\frac{1}{3}}$$

When $r = \left(\frac{15}{\pi}\right)^{\frac{1}{3}}$, $\frac{dV}{dt} = \lambda$.

$$\lambda = 4\pi\left(\frac{15}{\pi}\right)^{\frac{2}{3}} \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{\lambda}{4\pi}\left(\frac{\pi}{15}\right)^{\frac{2}{3}}$$

Surface Area,

$$A = 4\pi r^2$$

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

When $r = \left(\frac{15}{\pi}\right)^{\frac{1}{3}}$, $\frac{dr}{dt} = \frac{\lambda}{4\pi}\left(\frac{\pi}{15}\right)^{\frac{2}{3}}$.

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$= 8\pi\left(\frac{15}{\pi}\right)^{\frac{1}{3}} \frac{\lambda}{4\pi}\left(\frac{\pi}{15}\right)^{\frac{2}{3}}$$

$$= 2\lambda\left(\frac{\pi}{15}\right)^{\frac{1}{3}} \text{ cm}^2/\text{s}$$

Q2

(i)

The height of triangle ADG is $\dfrac{a}{\tan\theta} = \dfrac{a}{t}$.

Hence $AH = 2a + \dfrac{a}{t} = a\left(2 + \dfrac{1}{t}\right)$.

$BH = BE + EH = 2a\tan\theta + a = a(2t+1)$

Area $S = \dfrac{1}{2}(AH)(BC)$

$S = \dfrac{a}{2}\left(2 + \dfrac{1}{t}\right)\left(2a(2t+1)\right)$

$S = a^2\left(2 + \dfrac{1}{t}\right)(2t+1)$

$S = a^2\left(4 + 4t + \dfrac{1}{t}\right)$

(ii)

$\dfrac{dS}{dt} = a^2\left(4 - \dfrac{1}{t^2}\right)$

When $\dfrac{dS}{dt} = 0$,

$t^2 = \dfrac{1}{4}$

$\Rightarrow \quad t = \pm\dfrac{1}{2}$

Reject $t = \tan\theta = -\frac{1}{2}$ as θ is acute

$$\frac{d^2S}{dt^2} = a^2\left(\frac{2}{t^3}\right)$$

When $t = \frac{1}{2}$, $\frac{d^2S}{dt^2} = a^3\left(\frac{2}{\left(\frac{1}{2}\right)^3}\right) = 16a > 0$.

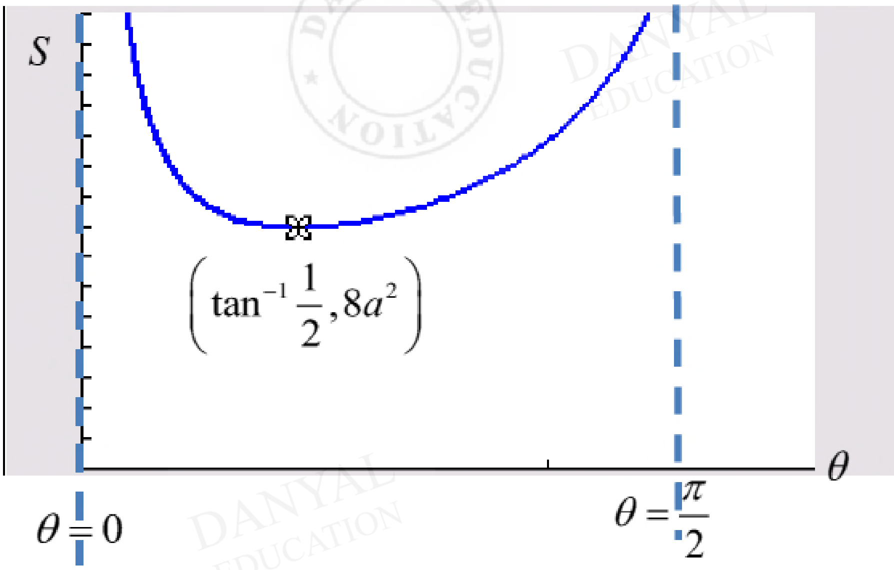
Hence the minimum value of S occurs when $t = \frac{1}{2}$.

Minimum $S = a^2(4+2+2) = 8a^2$.

(iii)

To sketch the graph of

$$S = a^2\left(4 + 4\tan\theta + \frac{1}{\tan\theta}\right)$$



Q3

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$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= \frac{(1+t)2t - t^2}{(1+t)^2} \div \frac{(1+t)(1) - t}{(1+t)^2}$$

$$= t^2 + 2t$$

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i

At point $\left(\frac{p}{1+p}, \frac{p^2}{1+p}\right)$, $t = p$

Equation of tangent at point $\left(\frac{p}{1+p}, \frac{p^2}{1+p}\right)$,

$$y - \frac{p^2}{1+p} = \left(p^2 + 2p\right)\left(x - \frac{p}{1+p}\right)$$

$$y = p(p+2)x + \frac{p^2}{1+p} - \frac{p^3}{1+p} - \frac{2p^2}{1+p}$$

$$y = p(p+2)x - \frac{p^2(p+1)}{1+p}$$

$$y = p(p+2)x - p^2$$

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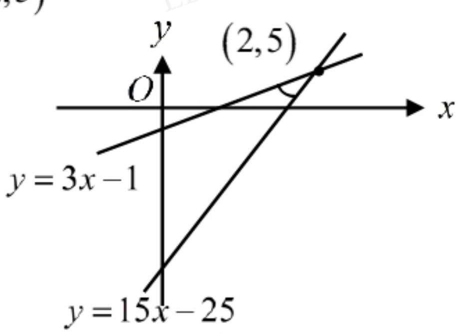
ii

Tangents pass through $(2,5)$

$$\Rightarrow 5 = p(p+2)(2) - p^2$$

$$p^2 + 4p - 5 = 0$$

$$p = -5 \quad \text{or} \quad p = 1$$



Equations of tangents are

$y = 3x - 1$ and $y = 15x - 25$

Required acute angle between the 2 tangents

$$= \tan^{-1}(15) - \tan^{-1}(3)$$

$$= 0.255 \text{ rad or } 14.6^\circ$$