A Level H2 Math

Differentiation and its Applications Test 7

Q1
It is given that a curve C has parametric equations

$$x = t^2 - t, \quad y = \frac{1}{t^2 + 1} \quad \text{for } -2 \le t < 2.$$

(i) Sketch C, indicating clearly the coordinates of the end points and the points where C cuts the y-axis. [4]

(ii) Find the equation of the tangent to C that is parallel to the y-axis. [4]

(iii) Express the area of the region bounded by C, the tangent found in part **(ii)** and both axes, in the form

$$\int_a^b \mathrm{f}(t)\, \mathrm{d}t,$$

where the function f and the constants a and b are to be determined. Hence find this area, leaving your answer in exact form. [5]

Q2
A right circular cone has base radius r cm and height h cm. As r and h vary, its curved surface area, $\pi r\sqrt{(r^2 + h^2)}$ cm^2, remains constant.

It is given that when $r = \sqrt{2}$ cm, the magnitude of the rate of change of h is 10 times the magnitude of the rate of change of r. Given also that $h > r$, find the height of the cone at this instant. [4]

Q3

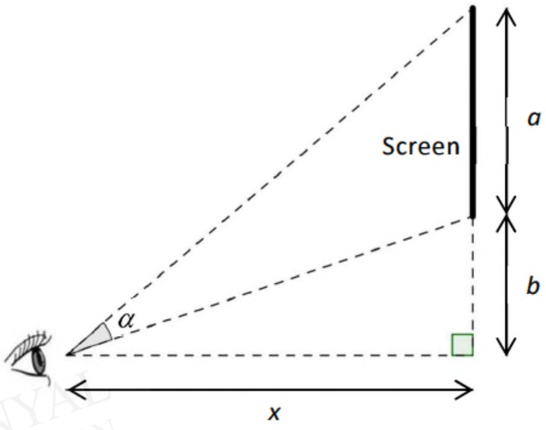


Fig. 1

Mr Tan is planning to set up a home theatre in his spacious rectangular living room. A projector screen with height a metres is to be positioned against one of the walls b metres above the eye level (see Fig. 1). He is trying to decide on the horizontal distance between the sofa and the screen so that the viewing angle α of the projection screen is as large as possible.

(i) Show that $\alpha = \tan^{-1}\dfrac{a+b}{x} - \tan^{-1}\dfrac{b}{x}$, where x is the horizontal distance between the sofa and the screen in metres. [1]

(ii) Use differentiation to show that the value of x which gives the maximum value of α satisfies the equation

$$\frac{a+b}{x^2+(a+b)^2} = \frac{b}{x^2+b^2}.$$

Solve for x and leave your answer in terms of a and b. [4]
[It is not necessary to verify the nature of the maximum point in this part.]

Mrs Tan proposed an alternative way of arrangement. She proposed to place the sofa against the wall opposite the screen, which is c metres away, and to vary the vertical position of the screen placed y metres above the eye level in order to maximise the angle α (see Fig. 2).

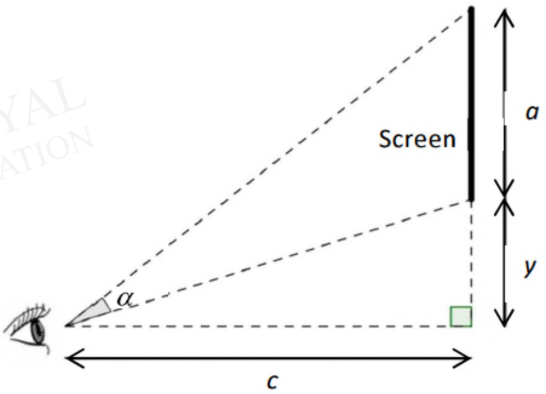


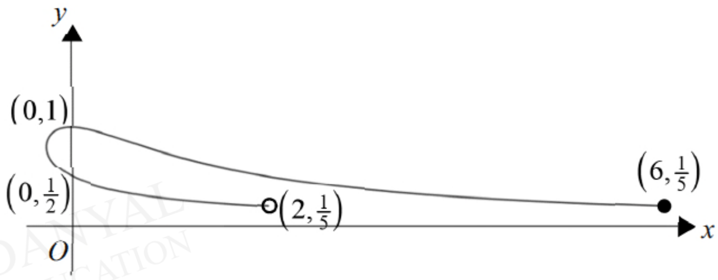
Fig. 2

(iii) Use differentiation to find the value of y which gives the maximum value of α, leaving your answer in terms of a. Interpret the answer in this context. [5]

Answers

Differentiation and its Applications Test 7

Q1



When $x=0,\ t(t-1)=0 \quad \Rightarrow t=0 \text{ or } t=1$

$\Rightarrow y=1 \text{ or } y=\frac{1}{2}$

Coordinates are $(0,1)$ and $\left(0,\frac{1}{2}\right)$.

(ii)

$\frac{dx}{dt}=2t-1,\quad \frac{dy}{dt}=\frac{-2t}{\left(t^2+1\right)^2}$

$\therefore \frac{dy}{dx}=\frac{-2t}{\left(t^2+1\right)^2}\times\frac{1}{2t-1}$

$=\frac{-2t}{\left(t^2+1\right)^2(2t-1)}$

When tangent is parallel to y-axis,

$\left(t^2+1\right)^2(2t-1)=0 \Rightarrow t=\frac{1}{2} \qquad \left(\because \left(t^2+1\right)^2>0\right)$

Equation of tangent: $x=-\frac{1}{4}$

(iii)

Area of the required region

$=\int_{-1/4}^{0} y \quad dx$

$=\int_{1/2}^{1} \frac{1}{t^2+1}(2t-1)\,dt$

$=\int_{1/2}^{1} \frac{2t}{t^2+1}-\frac{1}{t^2+1}\,dt$

$=\left[\ln\left(t^2+1\right)-\tan^{-1}t\right]_{1/2}^{1}$

$=\left[\left(\ln 2-\frac{\pi}{4}\right)-\left(\ln\frac{5}{4}-\tan^{-1}\frac{1}{2}\right)\right]$

$=\ln\frac{8}{5}-\frac{\pi}{4}+\tan^{-1}\frac{1}{2}$

When $x=-\frac{1}{4},\ t=\frac{1}{2}$ When $x=0,\ t=1$

Q2

Let $A=\pi r\sqrt{r^2+h^2}$.

$\Rightarrow \quad A^2=\pi^2 r^2\left(r^2+h^2\right)$

Differentiate w.r.t. t :

$$r^2\left(2r\frac{\mathrm{d}r}{\mathrm{d}t}+2h\frac{\mathrm{d}h}{\mathrm{d}t}\right)+\left(r^2+h^2\right)\left(2r\frac{\mathrm{d}r}{\mathrm{d}t}\right)=0$$

(Note: $\frac{\mathrm{d}A}{\mathrm{d}t}=0$ since A is a constant)

Since $r\neq 0$, $\left(2r^2+h^2\right)\frac{\mathrm{d}r}{\mathrm{d}t}+hr\frac{\mathrm{d}h}{\mathrm{d}t}=0$

$$\Rightarrow \quad \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)\div\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)=\frac{2r^2+h^2}{-hr}$$

When $r=\sqrt{2}$, $\frac{\mathrm{d}h}{\mathrm{d}t}=-10\frac{\mathrm{d}r}{\mathrm{d}t} \Rightarrow \frac{4+h^2}{-\sqrt{2}h}=-10$

$$\Rightarrow h^2-10\sqrt{2}h+4=0$$

Solving: $h=13.9$ (3sf) or $h=0.289$ (3sf)

Since $h>r$, the height of the cone required is 13.9 cm (to 3 sf).

Q3

(i) Let β be the angle of elevation of the bottom of the screen from eye-level.

$$\tan(\alpha+\beta)=\frac{a+b}{x}\Rightarrow \alpha+\beta=\tan^{-1}\frac{a+b}{x}$$

$$\tan(\beta)=\frac{b}{x}\Rightarrow \beta=\tan^{-1}\frac{b}{x}$$

$$\alpha=(\alpha+\beta)-\beta=\tan^{-1}\frac{a+b}{x}-\tan^{-1}\frac{b}{x}$$

(ii)

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x}=\frac{1}{1+\left(\frac{a+b}{x}\right)^2}\left(-\frac{a+b}{x^2}\right)-\frac{1}{1+\left(\frac{b}{x}\right)^2}\left(-\frac{b}{x^2}\right)$$

$$=-\frac{a+b}{x^2+(a+b)^2}+\frac{b}{x^2+b^2}$$

For maximum α: $\frac{\mathrm{d}\alpha}{\mathrm{d}x}=-\frac{(a+b)}{x^2+(a+b)^2}+\frac{b}{x^2+b^2}=0$

$$\frac{(a+b)}{x^2+(a+b)^2}=\frac{b}{x^2+b^2}\quad \text{(Shown)}$$

$$(a+b)(x^2+b^2)=b[x^2+(a+b)^2]$$

$$(a+b)x^2+(a+b)b^2=bx^2+b(a+b)^2$$

$$ax^2=b(a+b)[a+b-b]=ab(a+b)$$

$$x=\sqrt{b(a+b)}\quad \text{or}\quad -\sqrt{b(a+b)}\ \text{(NA since } x>0)$$

(iii) Let the screen be positioned y metres above the eye level.

$$\alpha=\tan^{-1}\frac{a+y}{c}-\tan^{-1}\frac{y}{c}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}y}=\frac{1}{1+\left(\frac{a+y}{c}\right)^2}\left(\frac{1}{c}\right)-\frac{1}{1+\left(\frac{y}{c}\right)^2}\left(\frac{1}{c}\right)$$

$$=\frac{c}{c^2+(a+y)^2}-\frac{c}{c^2+y^2}$$

$$=\frac{c\left[c^2+y^2-c^2-(a+y)^2\right]}{\left[c^2+(a+y)^2\right](c^2+y^2)}$$

$$=\frac{c\left[y+(a+y)\right]\left[y-(a+y)\right]}{\left[c^2+(a+y)^2\right](c^2+y^2)}$$

$$=\frac{-ac\left[a+2y\right]}{\left[c^2+(a+y)^2\right](c^2+y^2)}\qquad =\frac{-2ac\left[y+a/2\right]}{\left[c^2+(a+y)^2\right](c^2+y^2)}$$

For maximum α : $\dfrac{\mathrm{d}\alpha}{\mathrm{d}y} = \dfrac{-ac\left[a+2y\right]}{\left[c^2+(a+y)^2\right]\left(c^2+y^2\right)} = 0$

$$\Rightarrow y = -\frac{a}{2} \text{(since } a \neq 0 \text{ and } c \neq 0\text{)}$$

y	$\left(-\frac{a}{2}\right)^-$	$\left(-\frac{a}{2}\right)$	$\left(-\frac{a}{2}\right)^+$
$\left[y+\frac{a}{2}\right]$	<0	0	>0
$-2ac\left[y+\frac{a}{2}\right]$, where $-2ac<0$	>0	0	<0
$\dfrac{\mathrm{d}\alpha}{\mathrm{d}y}$	>0	0	<0

Therefore $y = -\dfrac{a}{2}$ gives the maximum viewing angle α .

Interpretation of the answer:
In order to maximise the viewing angle α , the centre of the screen need to be placed at eye level regardless of the position of the sofa.