A Level H2 Math

Differentiation and its Applications Test 6

Q1

A particle is moving along a curve, C, such that its position at time t seconds after it is set into motion is given by the parametric equations

$$x = t + e^{-2t}, \ y = t - e^{-2t}.$$

(i) State the coordinates of the initial position of the particle. [1]

(ii) Explain what would happen to the path of the particle after a long time. [1]

At the time of 2 seconds after the particle was set into motion, an external force struck the particle resulting in the particle moving in a straight line along the normal to the path at the point of collision.

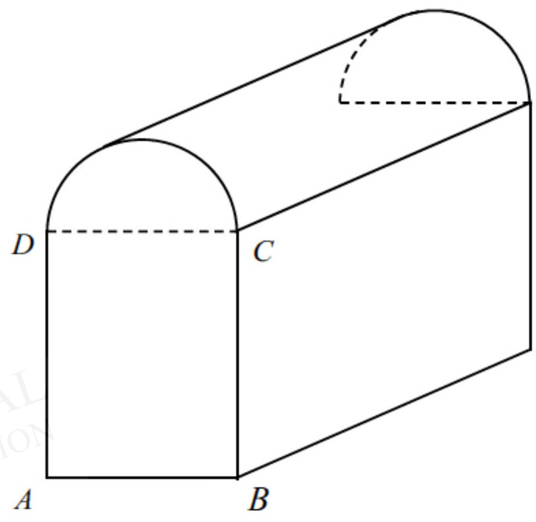
(iii) Find an equation for the normal to the curve C at the point $t = 2$, leaving your answer correct to 3 decimal places. [3]

After T seconds, where $T > 2$, the particle reaches point A, which lies on the x-axis, and stops moving.

(iv) Find the coordinates of the point A. Hence, give a sketch of the path traced by the particle, indicating the coordinates of any axial intercepts. [4]

(v) Find the total area bounded by the path of the particle in the first T seconds and the positive x-axis. [4]

Q2



A heavy wooden chest has a cross-sectional area made up of a rectangle and a semi-circle as shown in the diagram above. The wooden chest is constructed such that the perimeter of the cross-sectional area is 100 cm. It is given that the wooden chest is $2(a + b)$ cm long and the lengths of AB and BC are $2a$ cm and $2b$ cm respectively, where $a < 70$.

(i) Express b in terms of a. [1]

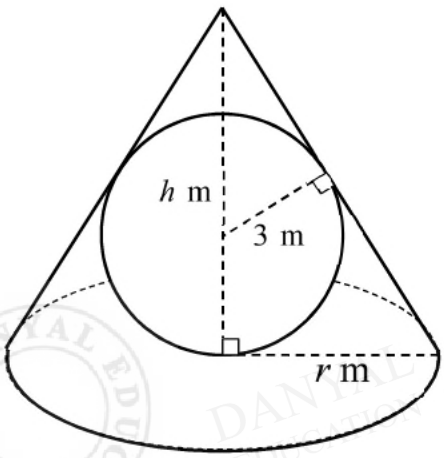
(ii) Show that the cross-sectional area of the wooden chest is given by $S = 100a - \frac{a^2}{2}(\pi + 4)$ and find the volume of the chest in terms of a and π. [4]

(iii) As a varies, find the value of a such that the volume of this wooden chest is greatest and find this volume correct to 2 decimal places. [5]

Q3

[It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$ and the volume and surface area of a sphere of radius r are $\frac{4}{3}\pi r^3$ and $4\pi r^2$ respectively.]

In a distant Northern kingdom of Drivenbell, Elsanna builds a spherical snowball with radius 3 m. The snowball is inscribed in a right conical container of base radius r m and height h m. The container is specially designed to allow the snowball to remain intact with fixed radius 3 m (see diagram).



(i) By considering the slant height of the cone, show that $r = \dfrac{3h}{\sqrt{\left(h^2 - 6h\right)}}$. [3]

(ii) Use differentiation to find the values of h and r that give a minimum volume for the container. Find the value of the minimum volume. [6]

The snowball is being removed from the container and it starts to melt under room temperature.

(iii) Assuming that the snowball remains spherical as it melts, find the rate of decrease of its volume at the instant when the radius of the sphere is 2.5 m, given that the surface area is decreasing at 0.75 m^2 per minute at this instant. [5]

<u>Answers</u>

<u>Differentiation and its Applications Test 6</u>

Q1

(i) At the original position, $t = 0$

$x = 0 + e^0 = 1$ and $y = 0 - e^0 = -1$

Thus the coordinates are $(1, -1)$.

(ii) As t tends to infinity, $e^{-2t} \to 0$ so $x \to t$ and $y \to t$

Thus, the path of the particle **approaches the line $y = x$**

(iii) $\dfrac{dy}{dt} = 1 + 2e^{-2t}$ and $\dfrac{dx}{dt} = 1 - 2e^{-2t}$

$\dfrac{dy}{dx} = \dfrac{1 + 2e^{-2t}}{1 - 2e^{-2t}}$

At $t = 2$, $x = 2 + e^{-4} = 2.01832$, $y = 2 - e^{-4} = 1.98168$ and $\dfrac{dy}{dx} = \dfrac{1 + 2e^{-4}}{1 - 2e^{-4}}$

Gradient of normal = $\dfrac{2e^{-4} - 1}{1 + 2e^{-4}} = -0.92933$

Thus, an equation for C_2 is $y - 1.98168 = -0.92933(x - 2.01832)$

i.e. $y = -0.92933x + 3.85737$

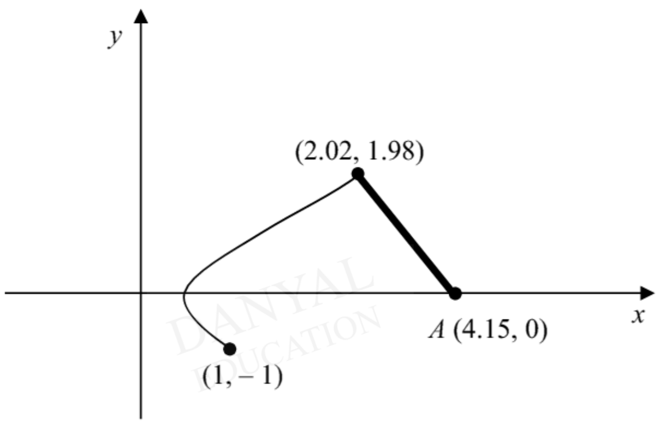
i.e. $y = -0.929x + 3.857$ (correct to 3 d.p.)

(iv) At point A, $y = 0$

$0 = -0.929x + 3.857 \Rightarrow x = 4.15178$

Coordinates of A are (4.15, 0)

Sketch of motion of particle:



(v) Consider the curve C_1 when $y = 0$,

$t = e^{-2t}$ and solving by GC, $t = 0.4263$

Thus, $x = 0.85261$

Required area

$$= \int_{0.852}^{2.02} y \, dx + \int_{2.02}^{4.15} (-0.929x + 3.857) \, dx$$

$$= \int_{0.4263}^{2} (t - e^{-2t})(1 - 2e^{-2t}) dt + \int_{2.02}^{4.15} (-0.929x + 3.857) \, dx$$

= 3.5576 units^2

= 3.56 units^2

Q2

(i) Perimeter of cross-sectional area $=100=(2a+4b)+\frac{1}{2}(2\pi a)$

$$\Rightarrow 100=4b+a(\pi+2)$$

$$\Rightarrow b=\frac{100-a(\pi+2)}{4}$$

(ii) $S=(2a)(2b)+\frac{1}{2}(\pi a^2)$

$$=4a\left[\frac{100-a(\pi+2)}{4}\right]+\frac{\pi}{2}a^2$$

$$=100a-a^2(\pi+2)+\frac{\pi}{2}a^2$$

$$=100a-\frac{a^2}{2}(2\pi+4-\pi)$$

$$=100a-\frac{a^2}{2}(\pi+4) \quad \text{(shown)}$$

Note that, $a+b=a+\frac{100-a(\pi+2)}{4}$

$$=\frac{4a+100-a(\pi+2)}{4}$$

$$=\frac{1}{4}\left[100+a(2-\pi)\right]$$

$$V=\left[100a-\frac{a^2}{2}(\pi+4)\right]2(a+b)$$

$$=\left[100a-\frac{a^2}{2}(\pi+4)\right]\cdot\frac{2}{4}\left[100+a(2-\pi)\right]$$

$$=\frac{a}{2}\left[100-\frac{a}{2}(\pi+4)\right]\cdot\left[100+a(2-\pi)\right]$$

$$=5000a-75\pi a-\frac{a^3}{4}(\pi^2+2\pi-8)$$

(iii) $\frac{\mathrm{d}V}{\mathrm{d}a}=5000-150\pi a-\frac{3}{4}a^2(\pi^2+2\pi-8)$

When $\frac{\mathrm{d}V}{\mathrm{d}a}=0$, using the GC, $a=12.70471$ or $a=64.36321$

For $a=12.70471$			
A	a^-	a	a^+
Sign	–	0	+
$\frac{\mathrm{d}V}{\mathrm{d}a}$	/	—	\

For $a=64.36321$			
a	a^-	a	a^+
sign	–	0	+
$\frac{\mathrm{d}V}{\mathrm{d}a}$	\	—	/

Thus when $a=12.70471=12.7$ (3 s.f.), volume is greatest.
Using the GC, greatest volume is 29671.95154=29671.95 cm^3.

Q3

(i)

Let l be the slant height of the cone.

$$l^2 = h^2 + r^2 \quad ----(1)$$

Using similar triangles,

$$\frac{h-3}{l} = \frac{3}{r}$$

$$l = \frac{rh-3r}{3} \quad ----(2)$$

Equating (1) and (2),

$$\left(\frac{rh-3r}{3}\right)^2 = h^2 + r^2 \quad ----(*)$$

$$r^2h^2 - 6r^2h + 9r^2 = 9h^2 + 9r^2$$

$$r^2(h^2 - 6h) = 9h^2$$

$$\therefore r = \frac{3h}{\sqrt{h^2-6h}} \qquad \text{(Since } r > 0\text{)}$$

(ii)

Volume of cone, $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(\frac{3h}{\sqrt{h^2-6h}}\right)^2 h$$

$$= \frac{3\pi h^3}{h^2-6h}$$

$$= \frac{3\pi h^2}{h-6}$$

$$\frac{dV}{dh} = \frac{6\pi h(h-6) - 3\pi h^2}{(h-6)^2}$$

$$= \frac{3\pi h^2 - 36\pi h}{(h-6)^2}$$

$$\frac{\mathrm{d}V}{\mathrm{d}h}=0 \qquad \Rightarrow \qquad 3\pi h^2-36\pi h=0$$

$$h(h-12)=0$$

$$h=12 \text{ or } h=0 \text{ (reject } \because h>0)$$

h	12^-	12	12^+
Sign of $\frac{\mathrm{d}V}{\mathrm{d}h}$	$-$ ve	0	+ ve
Tangent	\	—	/

Thus, V is a minimum when $h=12$

When $h=12$,

$$r=\frac{3(12)}{\sqrt{(12)^2-6(12)}}=\frac{6}{\sqrt{2}} \qquad (\approx 4.2426)$$

$$V=\frac{3\pi(12)^2}{12-6}=72\pi \qquad (\approx 226.195)$$

(iii)

Let R be the radius of the snowball

$$S=4\pi R^2 \qquad \Rightarrow \qquad \frac{\mathrm{d}S}{\mathrm{d}t}=8\pi R\frac{\mathrm{d}R}{\mathrm{d}t}$$

$$V=\frac{4}{3}\pi R^3 \qquad \Rightarrow \qquad \frac{\mathrm{d}V}{\mathrm{d}t}=4\pi R^2\frac{\mathrm{d}R}{\mathrm{d}t}$$

When $R=2.5$, $\frac{\mathrm{d}S}{\mathrm{d}t}=-0.75 \quad \Rightarrow \quad 8\pi(2.5)\frac{\mathrm{d}R}{\mathrm{d}t}=-0.75$

$$\frac{\mathrm{d}R}{\mathrm{d}t}=-\frac{3}{80\pi} \quad or \quad -\frac{0.0375}{\pi} \quad or \quad -0.0119366$$

$$\frac{\mathrm{d}V}{\mathrm{d}t}=4\pi(2.5)^2\left(-\frac{3}{80\pi}\right)=-\frac{15}{16} \quad or \quad -0.9375$$

At the instant when $R=2.5$ m, the rate of decrease of volume is 0.9375 m^3 per minute.