A Level H2 Math

Differentiation and its Applications Test 5

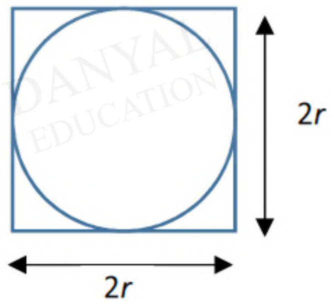
Q1

(a) When a liquid is poured onto a flat surface, a circular patch is formed. The area of the circular patch is expanding at a constant rate of $6\pi \text{ cm}^2/\text{s}$.

(i) Find the rate of change of the radius 24 seconds after the liquid is being poured. [3]

(ii) Explain whether the rate of change of the radius will increase or decrease as time passes. [1]

(b) A cylindrical can of volume 355 cm^3 with height h cm and base radius r cm is made from 3 pieces of metal. The curved surface of the can is formed by bending a rectangular sheet of metal, assuming that no metal is wasted in creating this surface. The top and bottom surfaces of the can are cut from square sheets of metal with length $2r$ cm, as shown below. The cost of the metal sheets is $\$K$ per cm^2.



(i) Show that the total cost of metal used, denoted by $\$C$, is given by

$$C = K\left(\frac{710}{r} + 8r^2\right).$$ [3]

(ii) Use differentiation to show that, when the cost of metal used is a minimum, then $\frac{h}{r} = \frac{8}{\pi}$. [5]

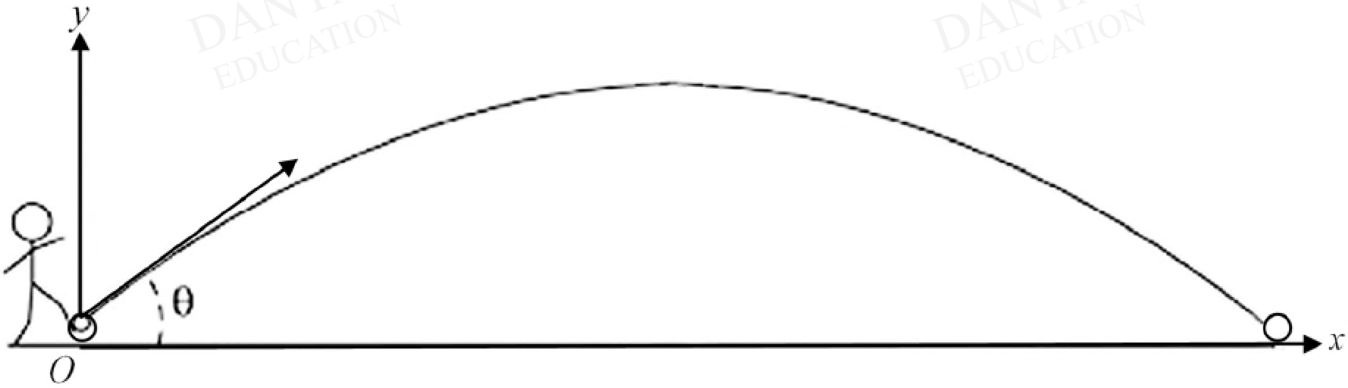
Q2

John kicked a ball at an acute angle θ made with the horizontal, and it moved in a projectile motion, as shown in the diagram. The initial velocity of the ball is u m s^{-1}. Taking John's position where he kicked the ball as the origin O, the ball's displacement curve is given by the parametric equations:

horizontal displacement, $x = ut\cos\theta,$

vertical displacement, $y = ut\sin\theta - 5t^2,$

where u and θ are constants and t is the time in seconds after the ball is kicked.



(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta - \frac{10}{u}t\sec\theta.$ [2]

(ii) If the initial velocity of the ball is 30 m s^{-1}, find the equation of the tangent to the displacement curve at the point where $t = \frac{1}{2}$, giving your answer in the form $y = (a\tan\theta + b\sec\theta)x + c$, where a, b and c are constants to be determined. [3]

Q3

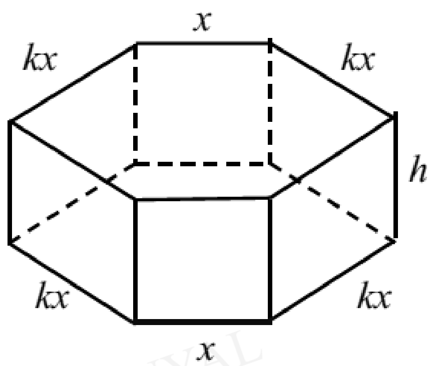


Fig 1

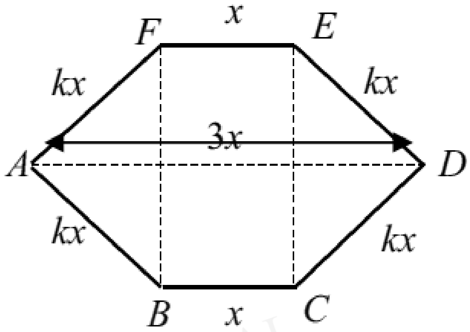


Fig 2

Figure 1 shows a solid metal hexagonal prism of height h cm. Figure 2 shows the hexagonal cross-section $ABCDEF$ of the prism where $AD = 3x$ cm, $BC = FE = x$ cm and the remaining 4 sides are of length kx cm each, where k is a constant.

Show that

$$S = 8x^2\sqrt{k^2 - 1} + 2xh(1 + 2k), \qquad [3]$$

where S is the surface area of this solid hexagonal prism.

(a) If the volume of the prism is fixed at 400 cm^3, use differentiation to find, in terms of k, the exact value of x that gives a stationary value of S. [3]

Let $k = 2$.

(b) The prism is heated and it expands in such a way that, at time t seconds, the rate of increase of x is the same as the rate of increase of its height h. At the instant when $x = 3$, the prism's height is 8 cm and its surface area is increasing at a constant rate of 0.5 cm^2/s. Find the rate of change of the volume of the prism at this instant. [6]

Answers

Differentiation and its Applications Test 5

Q1

(a)(i) Let A cm^2 be area of the circular patch.

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$

Given $\frac{dA}{dt} = 6\pi$ cm^2/s, a **constant**

This means that, in 1 s, A increases by 6π cm^2 **constantly**.

When $t = 0$, $A = 0$

When $t = 24$, $A = 24 \times 6\pi = 144\pi$

$$\pi r^2 = 144\pi$$
$$r = 12 \quad \text{(reject } r = -12 \text{ since } r > 0\text{)}$$
$$\frac{dA}{dr} = 2\pi(12) = 24\pi$$
$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$
$$6\pi = 24\pi \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1}{4}$$

$\therefore$ rate of change of the radius is $\frac{1}{4}$ cm/ s.

(a)(ii)

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$
$$6\pi = 2\pi r \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{6\pi}{2\pi r} = \frac{3}{r}$$

Method 1

As r increases, $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{r}$ decreases, $\therefore \frac{\mathrm{d}r}{\mathrm{d}t}$ will decrease as time passes.

Method 2

$$\begin{aligned}\frac{\mathrm{d}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)}{\mathrm{d}t} &= \frac{\mathrm{d}\left(\frac{3}{r}\right)}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} \\ &= \frac{-3}{r^2}\left(\frac{3}{r}\right) = \frac{-9}{r^3} < 0\end{aligned}$$

$\therefore \frac{\mathrm{d}r}{\mathrm{d}t}$ will decrease as time passes.

(b)(i)

$$\begin{aligned}V &= \pi r^2 h \\ 355 &= \pi r^2 h \\ \pi r h &= \frac{355}{r} \\ C &= K(2\pi r h) + 2K(4r^2) \\ &= K\left[2\left(\frac{355}{r}\right) + 8r^2\right] \\ &= K\left(\frac{710}{r} + 8r^2\right) \quad \text{(Shown)}\end{aligned}$$

(b)(ii) $\frac{\mathrm{d}C}{\mathrm{d}r} = \left(-\frac{710}{r^2}+16r\right)K$

For C to be a minimum, $\frac{\mathrm{d}C}{\mathrm{d}r}=0$.

$$-\frac{710}{r^2}+16r = 0$$
$$-710+16r^3 = 0$$
$$r^3 = \frac{355}{8}$$
$$r = \sqrt[3]{\frac{355}{8}} = 3.54 \text{ (3 sf)}$$
$$\frac{\mathrm{d}^2C}{\mathrm{d}r^2} = \left(\frac{1420}{r^3}+16\right)K = \left(\frac{1420}{\frac{355}{8}}+16\right)K = 48K > 0$$

Or

r	3.5	$\sqrt[3]{\frac{355}{8}} \approx 3.54$	3.6
$\frac{\mathrm{d}C}{\mathrm{d}r}$	$-1.96K < 0$	0	$2.82K > 0$

So, $r = \sqrt[3]{\frac{355}{8}}$ does give the minimum cost.

Recall $355 = \pi r^2 h$

$$h = \frac{355}{\pi r^2}$$
$$\therefore \frac{h}{r} = \frac{355}{\pi r^3} = \frac{355}{\pi\left(\frac{355}{8}\right)}$$
$$= \frac{8}{\pi} \quad \text{(Shown)}$$

Q2

(i)

$$\frac{dx}{dt} = u\cos\theta, \quad \frac{dy}{dt} = u\sin\theta - 10t,$$

$$\frac{dy}{dx} = \frac{u\sin\theta - 10t}{u\cos\theta}$$

$$= \tan\theta - \frac{10t}{u\cos\theta}$$

$$= \tan\theta - \frac{10}{u}t\sec\theta \quad \text{(Shown)}$$

(ii) When $u = 30$ and $t = \frac{1}{2}$,

$$x = 15\cos\theta, \ y = 15\sin\theta - \frac{5}{4}, \ \frac{dy}{dx} = \tan\theta - \frac{1}{6}\sec\theta$$

Equation of tangent is

$$y - 15\sin\theta + \frac{5}{4} = \left(\tan\theta - \frac{1}{6}\sec\theta\right)(x - 15\cos\theta)$$

$$= \left(\tan\theta - \frac{1}{6}\sec\theta\right)x - 15\sin\theta + \frac{5}{2}$$

$$\therefore y = \left(\tan\theta - \frac{1}{6}\sec\theta\right)x + \frac{5}{4}$$

Q3

$FB = EC = 2\sqrt{(kx)^2 - x^2} = 2x\sqrt{k^2 - 1}$

Area of cross-section of prism

$= \text{Area of } ABCD + \text{Area of } AFED$

$= 2(\text{Area of trapezium } ABCD)$

$= 2\left[\frac{1}{2}(x + 3x)\sqrt{(kx)^2 - x^2}\right]$

$= 4x^2\sqrt{k^2 - 1}$

Surface area of prism, $S = 2\left(4x^2\sqrt{k^2 - 1}\right) + 2xh + 4kxh$

Hence $S = 8x^2\sqrt{k^2 - 1} + 2xh(1 + 2k)$ (shown) --- (2)

a

Volume of prism $= 400 = \left(4x^2\sqrt{k^2 - 1}\right)h$

$h = \dfrac{100}{x^2\sqrt{k^2 - 1}}$ --- (1)

(1) in (2): $S = 8x^2\sqrt{k^2 - 1} + 2(1 + 2k)\left(\dfrac{100}{x\sqrt{k^2 - 1}}\right)$

$\dfrac{dS}{dx} = 16x\sqrt{k^2 - 1} - \dfrac{200(1 + 2k)}{x^2\sqrt{k^2 - 1}}$

When $\dfrac{dS}{dx} = 0$, $x^3 = \dfrac{200(1 + 2k)}{16(k^2 - 1)} \Rightarrow x = \sqrt[3]{\dfrac{25(1 + 2k)}{2(k^2 - 1)}}$

b When $k = 2$,

$S = 8x^2\sqrt{k^2-1} + 2xh(1+2k) = 8\sqrt{3}x^2 + 10xh$ and

$V = \left(4x^2\sqrt{k^2-1}\right)h = 4\sqrt{3}x^2h$

Given that $\frac{dx}{dt} = \frac{dh}{dt}$

$$\Rightarrow \frac{dh}{dx} = \frac{dh}{dt} \times \frac{dt}{dx} = 1$$

<u>Method 1</u>

$\frac{dS}{dx} = 8\sqrt{3}(2x) + 10h + 10x\frac{dh}{dx} = 16\sqrt{3}x + 10h + 10x$ --- (1)

$\frac{dV}{dx} = 4\sqrt{3}\left(h.2x + x^2\frac{dh}{dx}\right) = 4\sqrt{3}\left(2xh + x^2\right)$ --- (2)

When $x = 3$, $h = 8$, $\frac{dS}{dt} = 0.5$,

$\frac{dS}{dx} = 16\sqrt{3}(3) + 10(8+3) = 48\sqrt{3} + 110$

$\frac{dV}{dx} = 4\sqrt{3}\left(2.3.8 + 3^2\right) = 228\sqrt{3}$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dS} \times \frac{dS}{dt}$$
$$= 228\sqrt{3} \times \frac{1}{48\sqrt{3}+110} \times 0.5$$
$$= 1.02 \text{ (to 3 s.f.)}$$

<u>Method 2</u>

$\frac{dS}{dt} = 8\sqrt{3}\left(2x\frac{dx}{dt}\right) + 10\left(h\frac{dx}{dt} + x\frac{dh}{dt}\right) = \left(16\sqrt{3}x + 10h + 10x\right)\frac{dx}{dt}$ --- (1)

And

$\frac{dV}{dt} = 4\sqrt{3}\left(h.2x\frac{dx}{dt} + x^2\frac{dh}{dt}\right) = 4\sqrt{3}\left(2xh + x^2\right)\frac{dx}{dt}$ --- (2)

When $x = 3$, $h = 8$, $\frac{dS}{dt} = 0.5$, using eqn (1) to find $\frac{dx}{dt}$

$$0.5 = \left(16\sqrt{3}(3) + 10(8) + 10(3)\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{0.5}{48\sqrt{3}+110} \approx 0.0025888$$

Sub into (2) to get $\frac{dV}{dt}$

$$\frac{dV}{dt} = 4\sqrt{3}\left(2.3.8 + 3^2\right)(0.0025888) = 1.022343317 \approx 1.02 \text{ (to 3 sf)}$$