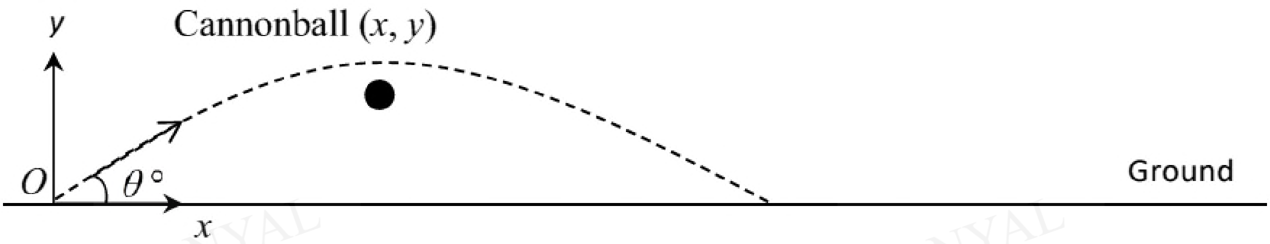
A Level H2 Math

Differentiation and its Applications Test 4

Q1



The diagram shows the trajectory of a cannonball fired off from an origin O with an initial speed of v ms^{-1} and at an angle of $\theta°$ above the ground. At time t seconds, the position of the cannonball can be modelled by the parametric equations

$$x = (v\cos\theta)t, \ \ y = (v\sin\theta)t - 5t^2,$$

where x m is the horizontal distance of the cannonball with respect to O and y m is the vertical distance of the cannonball with respect to ground level.

(i) Find the horizontal distance, d m, that a cannonball would have travelled by the time it hits the ground. Leave your answer in terms of v and θ. [4]

Use $v = 200$ to answer the remaining parts of the question.

An approaching target is travelling at a constant speed of 10 ms^{-1} along the ground. A cannonball is fired towards the target when it is 3000 m away. You may assume the height of the moving target is negligible.

(ii) Show that in order to hit the target, the possible angles at which the cannonball should be fired are $22.7°$ and $69.5°$. [2]

(iii) Explain at which angle the cannonball should be fired in order to hit the target earlier. [2]

(iv) Given that $\theta = 22.7$, find the angle that the tangent to the trajectory makes with the horizontal when $x = 370$. [4]

Q2

The variables y and x satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-\ln x}{x\ln x + 2x^2}.$$

(i) Show that the substitution $u = \frac{\ln x}{x}$ reduces the differential equation to $\frac{\mathrm{d}u}{\mathrm{d}y} = u + 2.$

Given that $y = 0$ when $x = 1$, show that $y = \ln\left(\frac{\ln x}{2x} + 1\right)$. [6]

The curve C has equation $y = \ln\left(\frac{\ln x}{2x} + 1\right)$. It is given that C has a maximum point and two asymptotes $y = a$ and $x = b$.

(ii) Find the exact coordinates of the maximum point. [2]

(iii) Explain why $a = 0$. [You may assume that as $x \to \infty$, $\frac{\ln x}{x} \to 0$.] [1]

(iv) Determine the value of b, giving your answer correct to 4 decimal places. [2]

(v) Sketch C. [2]

Q3

A particle moving along a path at time t, where $0 < t < \frac{\pi}{3}$, is defined parametrically by

$$x = \cot 3t \quad \text{and} \quad y = 2\operatorname{cosec} 3t + 1.$$

(a) The tangent to the path at the point $P(\cot 3p,\ 2\operatorname{cosec}3p + 1)$ meets the y-axis at the point Q. Show that the coordinates of Q is $(0,\ 2\sin 3p + 1)$. [4]

(b) The distance of the particle from the point $R(0, 1)$ is denoted by s, where $s^2 = x^2 + (y-1)^2$. Find the exact rate of change of the particle's distance from R at time $t = \frac{\pi}{4}$. [4]

Answers

Differentiation and its Applications Test 4

Q1

(i) To determine range of cannonball, we consider $y = 0$:

$0 = (v\sin\theta)t - 5t^2$

$0 = t[v\sin\theta - 5t]$

$\therefore t = 0$ (rejected) or $v\sin\theta - 5t = 0$

$\therefore t = \dfrac{v\sin\theta}{5}$

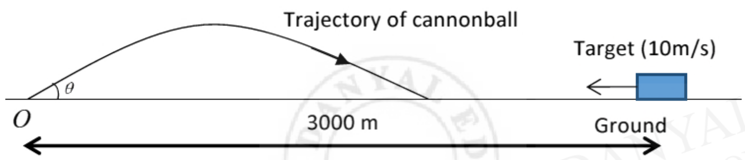
When $t = \dfrac{v\sin\theta}{5}$,

$x = (v\cos\theta)t$

$= (v\cos\theta)\dfrac{v\sin\theta}{5}$

$= \dfrac{v^2\sin\theta\cos\theta}{5}$ $\quad \therefore d = \dfrac{v^2\sin\theta\cos\theta}{5}$

(ii)



Time taken for cannonball to hit the ground = time taken for the target to reach the point of impact of the cannonball.

$\dfrac{v\sin\theta}{5} = \dfrac{3000 - d}{10}$

$2v\sin\theta = 3000 - \dfrac{v^2\sin\theta\cos\theta}{5}$

$\dfrac{(200)^2\sin\theta\cos\theta}{5} + 400\sin\theta = 3000$

Possible angles are $22.7°$ (to 1 dp) or $69.5°$ (to 1 dp). (shown)

(iii) Since $t = \dfrac{v\sin\theta}{5}$ when cannon hits target and $\dfrac{v\sin 22.7°}{5} < \dfrac{v\sin 69.5°}{5}$

Therefore to hit target earlier, cannonball should be fired at $22.7°$.

(iv) $\quad x = (200\cos 22.7°)t \qquad y = (200\sin 22.7°)t - 5t^2$

$\dfrac{dx}{dt} = 184.51 \qquad \dfrac{dy}{dt} = 77.181 - 10t$

$\therefore \dfrac{dy}{dx} = \dfrac{77.181 - 10t}{184.51}$

When $x = 370$, $\quad 184.51t = 370 \Rightarrow t = 2.0053$

$\therefore \dfrac{dy}{dx} = \dfrac{77.181 - 10(2.0053)}{184.51} = 0.30962$

Let the required angle be α.

$\tan\alpha = 0.30962 \Rightarrow \alpha = 17.2°$ (to 1dp)

Q2

(i) $u=\dfrac{\ln x}{x} \Rightarrow \dfrac{du}{dx}=\dfrac{1-\ln x}{x^2}$

$$\frac{du}{dy}=\frac{du}{dx}\times\frac{dx}{dy}=\frac{1-\ln x}{x^2}\times\frac{x\ln x+2x^2}{1-\ln x}=\frac{\ln x+2x}{x}$$

$$\frac{du}{dy}=u+2 \quad \text{(shown)}$$

$$\frac{1}{u+2}\frac{du}{dy}=1 \Rightarrow \ln|u+2|=y+c, \ c \text{ is an arbitrary constant}$$

$$|u+2|=e^{y+c}=e^c e^y$$

$$u+2=Ae^y, \ A \text{ is an arbitrary constant}$$

$$\frac{\ln x}{x}+2=Ae^y$$

$y=0, x=1: \quad A=2$

$$\frac{\ln x}{2x}+1=e^y$$

$$y=\ln\left(\frac{\ln x}{2x}+1\right) \quad \text{(shown)}$$

<u>Alternative</u>

$$\frac{1}{u+2}\frac{du}{dy}=1 \Rightarrow \ln|u+2|=y+c, \ c \text{ is an arbitrary constant}$$

With the boundary condition $u=0, y=0,$ we see that $u+2>0$

Thus $\ln|u+2|=y+c$ and $c=\ln 2$

(ii) $\dfrac{dy}{dx}=\dfrac{1-\ln x}{x\ln x+2x^2}$

When $\frac{dy}{dx}=0$, $1-\ln x=0 \quad \Rightarrow x=e$, $y=\ln\left(\frac{1}{2e}+1\right)$

Therefore the maximum point is $\left(e, \ln\left(\frac{1}{2e}+1\right)\right)$.

(iii) $y=\ln\left(\frac{\ln x}{2x}+1\right)$

When $x\to\infty, \frac{\ln x}{2x}\to 0.$

$y=\ln\left(\frac{\ln x}{2x}+1\right)\to \ln 1=0.$

Thus $a=0$ (shown)

(iv) For $y\to-\infty$, $\frac{\ln x}{2x}+1\to 0$

$\ln x+2x\to 0$

$x\to 0.4263$

$\therefore b=0.4263$

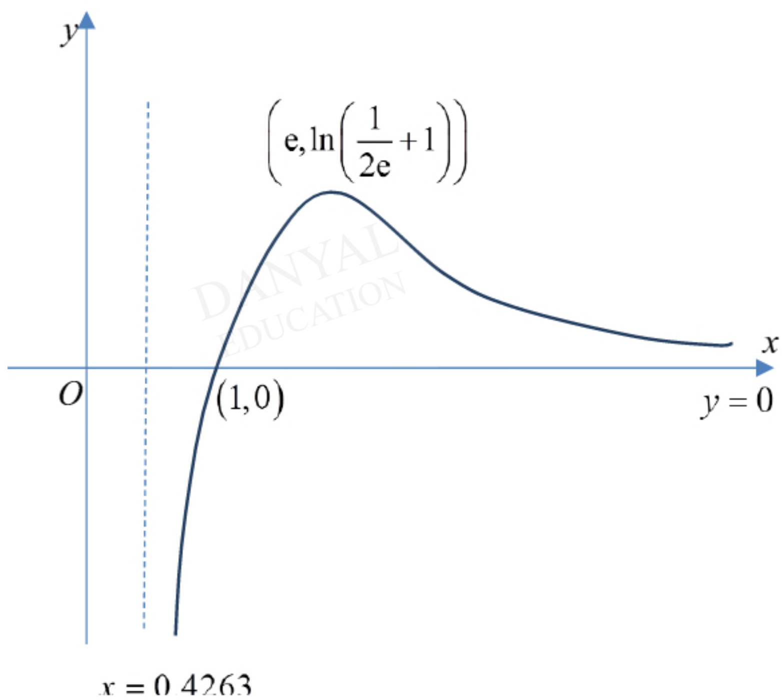
Alternative

When y is undefined, $\frac{dy}{dx}=\frac{1-\ln x}{x\ln x+2x^2}$ is undefined.

Thus $x\ln x+2x^2=0$.

Since $x>0$ for $\ln x$ to be defined, $\ln x+2x=0$.

(v)



Q3

(a)

$x = \cot 3t \;\Rightarrow\; \dfrac{dx}{dt} = -3\operatorname{cosec}^2 3t$

$y = 2\operatorname{cosec} 3t + 1 \;\Rightarrow\; \dfrac{dy}{dt} = -6\operatorname{cosec} 3t \cot 3t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-6\operatorname{cosec} 3t \cot 3t}{-3\operatorname{cosec}^2 3t}$$

$$= \frac{2\cot 3t}{\operatorname{cosec} 3t}$$

$$= 2\cos 3t$$

At point P, $\dfrac{dy}{dx}\Big|_{t=p} = 2\cos 3p$

Equation of tangent at P:

$y - (2\operatorname{cosec} 3p + 1) = 2\cos 3p(x - \cot 3p)$

When tangent meets y-axis, $x = 0$.

Hence $y = -(2\cos 3p)(\cot 3p) + (2\operatorname{cosec} 3p + 1)$

$$y = \frac{-2(\cos^2 3p)}{\sin 3p} + \frac{2}{\sin 3p} + 1$$

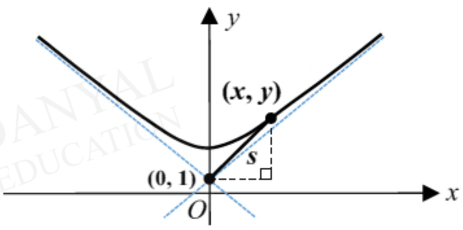
$$y=\frac{-2(\cos^2 3p-1)}{\sin 3p}+1$$

$$y=\frac{-2(-\sin^2 3p)}{\sin 3p}+1$$

$$y=2\sin 3p+1$$

Hence the coordinates of Q is $\left(0,\, 2\sin 3p+1\right)$. (shown)

(b)



Method 1

$$s^2=x^2+\left(y-1\right)^2$$

$$=\cot^2 3t+\left(2\operatorname{cosec}3t+1-1\right)^2$$

$$=(\operatorname{cosec}^2 3t-1)+4\operatorname{cosec}^2 3t$$

$$=5\operatorname{cosec}^2 3t-1$$

Differentiate w.r.t. t,

$$2s\frac{\mathrm{d}s}{\mathrm{d}t}=10\operatorname{cosec}3t(-\operatorname{cosec}3t\cot 3t)(3)$$

$$=-30\operatorname{cosec}^2 3t\cot 3t$$

$$s\frac{\mathrm{d}s}{\mathrm{d}t}=-15\operatorname{cosec}^2 3t\cot 3t$$

When $t=\frac{\pi}{4}$, $s^2=(2\sqrt{2}+1-1)^2+(-1)^2=9$

$\therefore\ s=3$ (since $s>0$)

$$\therefore\ \frac{\mathrm{d}s}{\mathrm{d}t}=-5\operatorname{cosec}^2 3\left(\tfrac{\pi}{4}\right)\cot 3\left(\tfrac{\pi}{4}\right)$$

$$=-5(2)(-1)$$

$$=10 \text{ unit/s}$$

Method 2

$$s^2=x^2+\left(y-1\right)^2$$

$$=\cot^2 3t+\left(2\operatorname{cosec}3t+1-1\right)^2$$

$$=\cot^2 3t+4\operatorname{cosec}^2 3t$$

Differentiate w.r.t. t,

$$2s\frac{\mathrm{d}s}{\mathrm{d}t}=2\cot 3t(-\operatorname{cosec}^2 3t)(3)+8\operatorname{cosec}3t(-\operatorname{cosec}3t\cot 3t)(3)$$

$$=-6\operatorname{cosec}^2 3t\cot 3t-24\operatorname{cosec}^2 3t\cot 3t$$

$$=-30\operatorname{cosec}^2 3t\cot 3t$$

$$s\frac{\mathrm{d}s}{\mathrm{d}t}=-15\operatorname{cosec}^2 3t\cot 3t$$

When $t=\frac{\pi}{4}$, $s^2=(2\sqrt{2}+1-1)^2+(-1)^2=9$

$\therefore\ s=3$ (since $s>0$)

$$\begin{aligned}\therefore\ \frac{\mathrm{d}s}{\mathrm{d}t}&=-5\operatorname{cosec}^2 3\left(\tfrac{\pi}{4}\right)\cot 3\left(\tfrac{\pi}{4}\right)\\&=-5(2)(-1)\\&=10\ \text{unit/s}\end{aligned}$$

<u>Method 3</u>

$s^2=x^2+(y-1)^2$

Differentiate w.r.t. t ,

$$2s\frac{\mathrm{d}s}{\mathrm{d}t}=2x\frac{\mathrm{d}x}{\mathrm{d}t}+2(y-1)\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$s\frac{\mathrm{d}s}{\mathrm{d}t}=x\frac{\mathrm{d}x}{\mathrm{d}t}+(y-1)\frac{\mathrm{d}y}{\mathrm{d}t}$$

When $t=\frac{\pi}{4}$,

$$x=\frac{1}{\tan\left(\frac{3\pi}{4}\right)}=-1,\qquad y=\frac{2}{\sin\left(\frac{3\pi}{4}\right)}+1=2\sqrt{2}+1$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-3\operatorname{cosec}^2 3t=\frac{-3}{\sin^2\left(\frac{3\pi}{4}\right)}=-6$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-6\cot 3t\operatorname{cosec}3t=\frac{-6}{\tan\left(\frac{3\pi}{4}\right)}\times\frac{1}{\sin\left(\frac{3\pi}{4}\right)}=6\sqrt{2}$$

$$s^2=(2\sqrt{2}+1-1)^2+(-1)^2=9$$

$\therefore\ s=3$ (since $s>0$)

$$\begin{aligned}\text{Hence }\frac{\mathrm{d}s}{\mathrm{d}t}&=\frac{1}{s}\left[x\frac{\mathrm{d}x}{\mathrm{d}t}+(y-1)\frac{\mathrm{d}y}{\mathrm{d}t}\right]\\&=\frac{1}{3}\left[(-1)(-6)+(2\sqrt{2})(6\sqrt{2})\right]\\&=10\ \text{unit/s}\end{aligned}$$