A Level H2 Math

Differentiation and its Applications Test 3

Q1

A curve C has parametric equations

$$x = \frac{4}{t+1} \quad \text{and} \quad y = t^2 - 3, \quad t \neq -1.$$

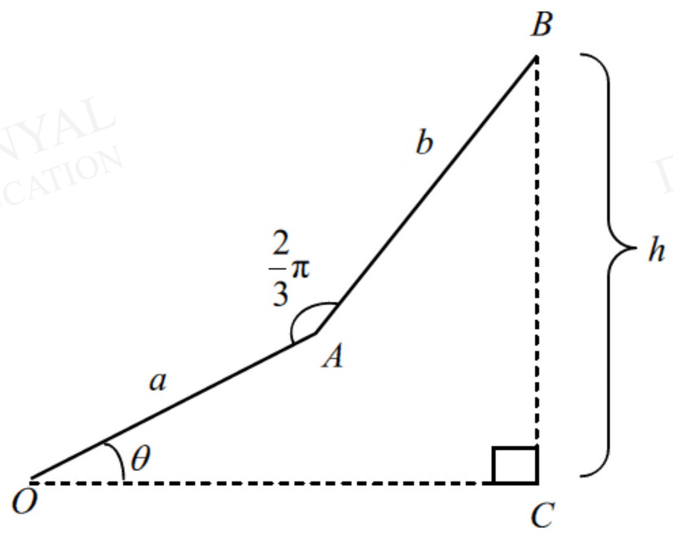
(i) Find $\frac{dy}{dx}$ in terms of t. [2]

(ii) Find the equation of the normal to C at P where $x = -2$. [3]

(iii) Find the other values of t where the normal at P meets the curve C again. [3]

Q2

The diagram below shows two adjoining lines OA and AB where $OA = a$ m, $AB = b$ m and obtuse angle OAB is $\frac{2}{3}\pi$. C is a point such that OC and CB are perpendicular to each other, $BC = h$ m, and angle AOC is θ where $0 < \theta < \frac{\pi}{6}$.



(i) Show that

$h = \sqrt{a^2 + ab + b^2}\sin(\theta + \alpha)$, where α is a constant to be determined in terms of a and b. [4]

It is given that $a = 1$ and $b = 2$.

(ii) Find the rate of change of θ when $\theta = \frac{\pi}{12}$ and h is decreasing at a rate of 0.5 m per minute. [3]

(iii) When θ is a sufficiently small angle, show that $h \approx p\theta^2 + q\theta + \sqrt{3}$, where constants p and q are to be determined exactly. [3]

Q3

(a) Given that $\operatorname{cosec} y = x$ for $0 < y < \frac{1}{2}\pi$, find $\frac{dy}{dx}$ in terms of y. Deduce that $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{(x^2-1)}}$ for $x > 1$. [3]

(b) The function f is such that $f(x)$ and $f'(x)$ exist for all real x. Sketch a possible graph of f which illustrates that the following statement is not necessarily true:

"If the equation $f'(x) = 0$ has exactly one root $x = 0$ and $f''(0) > 0$, then $f(x) \to \infty$ as $x \to \pm\infty$." [2]

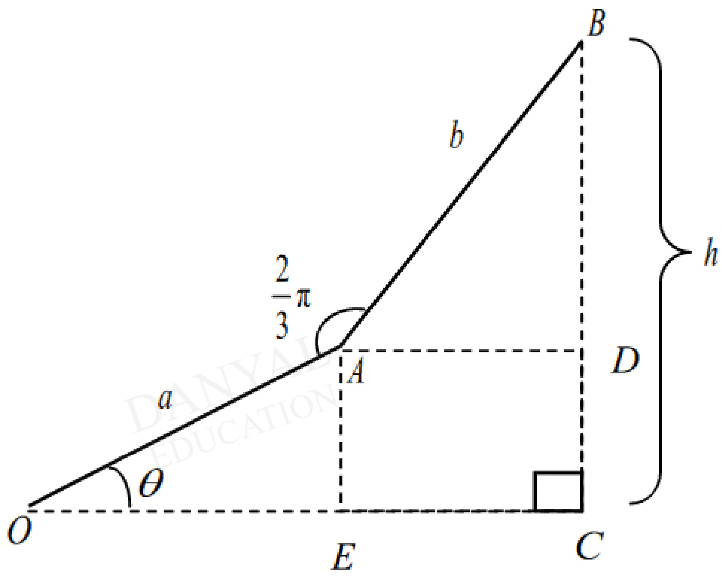
Answers

Differentiation and its Applications Test 3

Q1

(i)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{2t}{\frac{-4}{(t+1)^2}}=-\frac{t(t+1)^2}{2}$	Most students able to get the correct answers. Those who were unable to do made careless mistake in dx/dt.
(ii)	When $x=-2$, $\frac{4}{t+1}=-2$ $t=-3$ $y=6$ Gradient of normal $=\frac{2}{-3(-3+1)^2}=-\frac{1}{6}$ Equation of normal at $P(-2,6)$ is $y-6=-\frac{1}{6}(x+2)$ $y=-\frac{1}{6}x+\frac{17}{3}$ or $6y+x=34$	Most students got the correct concept to solve for the eqn of normal but lost the accuracy mark due of the wrong expression in part (i).
(iii)	$t^2-3=-\frac{1}{6}\left(\frac{4}{t+1}\right)+\frac{17}{3}$ $6(t+1)(t^2-3)=-4+34(t+1)$ $3(t+1)(t^2-3)=17t+15$ $3t^3+3t^2-9t-9=17t+15$ $3t^3+3t^2-26t-24=0$ Using GC, $t=-3$ (given) or $t=-0.915$ (3 s.f.) or $t=2.91$ (3 s.f.)	Many students attempt to convert the parametric eqn of the curve to cartesian form first then solve for the x values, then solve for the t values which lead to a longer method. Please note that this method may not work for all questions as it may be hard/impossible to convert to cartesian form. Also, many students didn't make use of their **GC** to solve and hence wasted their time to solve algebraically.

Q2



$h = BD + DC,\ DC = a\sin\theta,\ BD = b\sin\angle BAD$

$\angle BAD + \angle DAE + \angle OAE = 2\pi - \dfrac{2\pi}{3} = \dfrac{4\pi}{3}$

$\Rightarrow \angle BAD + \dfrac{\pi}{2} + \left(\dfrac{\pi}{2} - \theta\right) = \dfrac{4\pi}{3}$

$\Rightarrow \angle BAD = \theta + \dfrac{\pi}{3}$

$\Rightarrow BD = b\sin\left(\theta + \dfrac{\pi}{3}\right)$

$\therefore h = a\sin\theta + b\sin\left(\theta + \dfrac{\pi}{3}\right)$

Using $\sin(A+B) = \sin A\cos B + \cos A\sin B$, we have

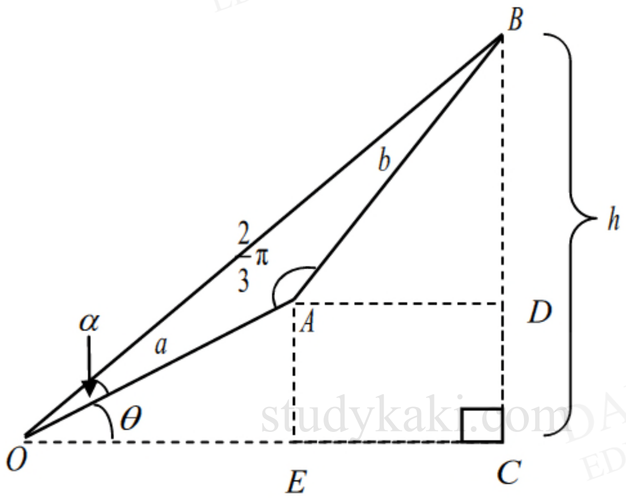
$$h = a\sin\theta + b\sin\left(\theta + \frac{\pi}{3}\right)$$
$$= a\sin\theta + b\sin\theta\cos\left(\frac{\pi}{3}\right) + b\cos\theta\sin\left(\frac{\pi}{3}\right)$$
$$= \left(a + \frac{b}{2}\right)\sin\theta + \frac{b\sqrt{3}}{2}\cos\theta$$
$$= R\sin(\theta + \alpha),$$

where $R=\sqrt{\left(a+\frac{b}{2}\right)^2+\left(\frac{\sqrt{3}}{2}b\right)^2}=\sqrt{a^2+ab+\frac{b^2}{4}+\frac{3b^2}{4}}=\sqrt{a^2+ab+b^2}$

$$\tan\alpha=\frac{\frac{\sqrt{3}}{2}b}{a+\frac{b}{2}}=\frac{\sqrt{3}b}{2a+b}\Rightarrow\alpha=\tan^{-1}\left(\frac{\sqrt{3}b}{2a+b}\right)$$

$$\therefore h=\sqrt{a^2+ab+b^2}\sin\left[\theta+\tan^{-1}\left(\frac{\sqrt{3}b}{2a+b}\right)\right]$$

<u>Method 2</u>



$$\sin(\theta+\alpha)=\frac{h}{OB}$$

$$OB^2=a^2+b^2-2ab\cos\frac{2\pi}{3}$$

$$OB=\sqrt{a^2+b^2+ab}$$

$$\frac{\sin\alpha}{b}=\frac{\sin\frac{2\pi}{3}}{\sqrt{a^2+b^2+ab}}$$

$$\sin\alpha=\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}$$

$$\alpha=\sin^{-1}\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}$$

$$h=\sqrt{a^2+ab+b^2}\sin\left(\theta+\sin^{-1}\frac{b\sqrt{3}}{2\sqrt{a^2+b^2+ab}}\right)$$

(ii)

Since $a=1, b=2, \alpha=\tan^{-1}\left(\frac{2\sqrt{3}}{2+2}\right)=\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$$h=\sqrt{7}\sin\left[\theta+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right], \frac{\mathrm{d}h}{\mathrm{d}\theta}=\sqrt{7}\cos\left[\theta+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

At $\theta=\frac{\pi}{12}$,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{\sqrt{7}\cos\left[\frac{\pi}{12}+\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]}\times(-0.5)$$

$=-0.337$ radians per minute

(iii)

$$\alpha=\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)\Rightarrow\sin\alpha=\sqrt{\frac{3}{7}}, \cos\alpha=\frac{2}{\sqrt{7}}$$

$$h=\left(a+\frac{b}{2}\right)\sin\theta+\frac{b\sqrt{3}}{2}\cos\theta$$

If θ is small,

$$\begin{aligned}h&=\sqrt{7}\sin(\theta+\alpha)\\&=\sqrt{7}\sin\theta\cos\alpha+\sqrt{7}\cos\theta\sin\alpha\\&\approx\sqrt{7}\theta\left(\frac{2}{\sqrt{7}}\right)+\sqrt{7}\left(1-\frac{\theta^2}{2}\right)\left(\sqrt{\frac{3}{7}}\right)\\&=2\theta+\sqrt{3}-\frac{\sqrt{3}\theta^2}{2}\\&=-\frac{\sqrt{3}\theta^2}{2}+2\theta+\sqrt{3}\end{aligned}$$

Q3

$\operatorname{cosec} y = x$

Diff wrt x:

$-\operatorname{cosec} y \cot y \dfrac{dy}{dx} = 1$

$\therefore \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{cosec} y \cot y}$

Using $\cot^2 y + 1 \equiv \operatorname{cosec}^2 y$, $\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{cosec} y\sqrt{(\operatorname{cosec} y)^2 - 1}}$

[since $0 < y < \dfrac{\pi}{2} \Rightarrow \tan y > 0$

$\Rightarrow \cot y > 0$

$\Rightarrow \cot y = \sqrt{(\operatorname{cosec} y)^2 - 1}$]

$= -\dfrac{1}{x\sqrt{x^2 - 1}}$ (shown)

Since $y = \operatorname{cosec}^{-1} x$,

$\dfrac{dy}{dx} = \dfrac{d}{dx}(\operatorname{cosec}^{-1} x) = -\dfrac{1}{x\sqrt{x^2 - 1}}$

Alternative

$\operatorname{cosec} y = x$

Diff wrt x :

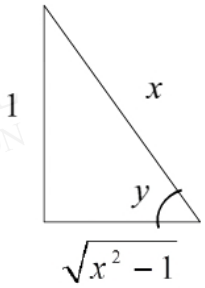
$-\operatorname{cosec} y \cot y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\operatorname{cosec} y \cot y}$

Since $\operatorname{cosec} y = x$,

$\therefore \frac{1}{\sin y} = x$

$\therefore \sin y = \frac{1}{x}$



By constructing the right angle triangle, $\tan y = \frac{1}{\sqrt{x^2 - 1}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\operatorname{cosec} y \cot y} = -\frac{\tan y}{\operatorname{cosec} y} = -\frac{1}{x\sqrt{x^2 - 1}}$ (shown)

(b)