A Level H2 Math

Differentiation and its Applications Test 2

Q1

(a) Using differentiation, find the exact dimensions of the rectangle of largest area that can be inscribed in the ellipse, $\frac{x^2}{9}+\frac{y^2}{36}=1$. Hence, find the area of this largest rectangle. [8]

(b) In the triangle DEF, angle $EDF=\frac{\pi}{3}$ and angle $DFE=\frac{\pi}{3}+\alpha$ and $EF=6$. Given that α is sufficiently small, show that

$$DF-DE\approx d\alpha,$$

where d is an exact constant to be determined. [5]

Q2

The curve C is defined parametrically by equations

$$x=\cos(p),\qquad y=\sin^3(p),\qquad 0\le p\le 2\pi$$

The point P on C has parameter p. Given that p is increasing at a rate of 0.5 units per second, find the rate at which $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing when $p=\frac{\pi}{3}$. [4]

Q3

A straight line passes through the point with coordinates $(4,\ 3)$, cuts the positive x-axis at point P and the positive y-axis at point Q. It is given that $\angle PQO=\theta$, where $0<\theta<\frac{\pi}{2}$ and O is the origin.

(i) Show that the equation of line PQ is given by $y=(4-x)\cot\theta+3$. [2]

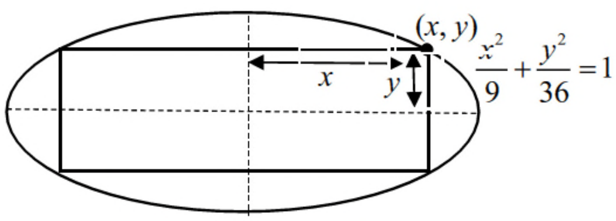
(ii) By finding an expression for $OP+OQ$, show that as θ varies, the stationary value of $OP+OQ$ is $a+b\sqrt{3}$, where a and b are constants to be determined. [5]

Answers

Differentiation and its Applications Test 2

Q1

(a)



Let (x, y) be a point on the ellipse.

Area of rectangle, A

$$=(2x)(2y)$$
$$=4xy$$
$$=4x\sqrt{36-4x^2}$$
$$=8\sqrt{9x^2-x^4}$$
$$=8\left(9x^2-x^4\right)^{\frac{1}{2}}$$

$$\frac{dA}{dx}=8\left(\frac{1}{2}\right)\left(9x^2-x^4\right)^{-\frac{1}{2}}\left(18x-4x^3\right)$$
$$=\frac{4\left(18x-4x^3\right)}{\sqrt{9x^2-x^4}}$$

When the area is the largest,

$$\frac{dA}{dx}=0$$
$$\frac{4\left(18x-4x^3\right)}{\sqrt{9x^2-x^4}}=0$$
$$18x-4x^3=0$$
$$2x\left(3-\sqrt{2}x\right)\left(3+\sqrt{2}x\right)=0$$

$x=0$ (rejected since $x\neq 0$)

or $x=\frac{3}{\sqrt{2}}$

or $x=-\frac{3}{\sqrt{2}}$ (rejected since $x>0$)

When $x=\frac{3}{\sqrt{2}}$, $y=3\sqrt{2}$

x	2.115	$\frac{3}{\sqrt{2}}\approx 2.12$	2.125
$\frac{dA}{dx}$	0.2013	0	-0.118
Slope	/	—	\

Area of the rectangle is a maximum
Maximum area

$= 8\left(9x^2 - x^4\right)^{\frac{1}{2}}$

$= 8\sqrt{9\left(\frac{3}{\sqrt{2}}\right)^2 - \left(\frac{3}{\sqrt{2}}\right)^4}$

$= 36 \text{ units}^2$

ALT

Note: $A = 8\left(9x^2 - x^4\right)^{\frac{1}{2}}$

Since $x, y > 0$, value of x that maximises A also maximises A^2

$A^2 = 64\left(9x^2 - x^4\right)$

$\frac{\mathrm{d}A^2}{\mathrm{d}x} = 64\left(18x - 4x^3\right) = 0$

$\Rightarrow x = \frac{3}{\sqrt{2}}$

(b)

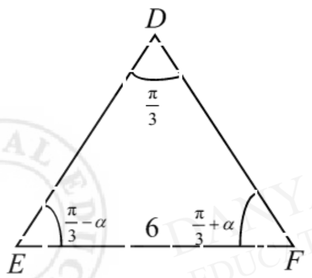
Using Sine rule,

$\frac{DF}{\sin\left(\frac{\pi}{3} - \alpha\right)} = \frac{6}{\sin\left(\frac{\pi}{3}\right)}$

$DF = 4\sqrt{3}\sin\left(\frac{\pi}{3} - \alpha\right)$

$\frac{DE}{\sin\left(\frac{\pi}{3} + \alpha\right)} = \frac{6}{\sin\left(\frac{\pi}{3}\right)}$

$DE = 4\sqrt{3}\sin\left(\frac{\pi}{3} + \alpha\right)$



$DF - DE$

$= 4\sqrt{3}\sin\left(\frac{\pi}{3} - \alpha\right) - 4\sqrt{3}\sin\left(\frac{\pi}{3} + \alpha\right)$

$= 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right) - 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha\right)$

$\approx 4\sqrt{3}\left[\frac{\sqrt{3}}{2}\left(1 - \frac{\alpha^2}{2}\right) - \frac{1}{2}\alpha - \frac{\sqrt{3}}{2}\left(1 - \frac{\alpha^2}{2}\right) - \frac{1}{2}\alpha\right]$

$= -4\sqrt{3}\alpha$

Q2

At point P, $x=\cos(p)$, $y=\sin^3(p)$

$$\frac{dy}{dp}=3\cos(p)\sin^2(p)$$

$$\frac{dx}{dp}=-\sin(p)$$

$$\frac{dy}{dx}=-3\sin(p)\cos(p)=\frac{-3}{2}\sin(2p)$$

Let $z=\dfrac{dy}{dx}$

$$\frac{dz}{dt}=\frac{dz}{dp}\cdot\frac{dp}{dt}$$

$$=-3\cos(2p)\cdot(0.5)$$

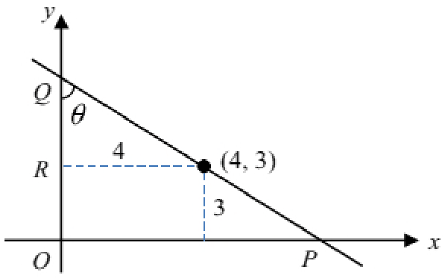
$$=\frac{-3}{2}\cos(2p)$$

$$\left.\frac{dz}{dt}\right|_{p=\frac{\pi}{3}}=\frac{-3}{2}\cos\left(\frac{2\pi}{3}\right)=0.75$$

Therefore, $\dfrac{dy}{dx}$ is increasing at 0.75 units per second when $p=\dfrac{\pi}{3}$.

Q3

(i)



Gradient $= -\dfrac{1}{\frac{OP}{OQ}} = -\dfrac{1}{\tan\theta} = -\cot\theta$

$\tan\theta = \dfrac{4}{QR} \Rightarrow QR = 4\cot\theta$

$y-$intercept $= 3 + 4\cot\theta$

Equation of line PQ is $y = -(\cot\theta)x + 3 + 4\cot\theta$

$y = (4-x)\cot\theta + 3$ (shown)

Poorly attempted. Many students could identify that they needed to find gradient but did not realized that gradient in this question is in fact negative. Students who attempted to 'work backwards' but did not show sufficient and accurate working were penalized.

(ii)

When $x = 0,\ y = 4\cot\theta + 3$

When $y = 0,\ 0 = (4-x)\cot\theta + 3$

$x = 4 - \dfrac{-3}{\cot\theta} = 4 + 3\tan\theta$

$OP + OQ = 4 + 3\tan\theta + 4\cot\theta + 3$

$= 7 + 3\tan\theta + 4\cot\theta$

Let $L = OP + OQ$

$\dfrac{dL}{d\theta} = 3\sec^2\theta - 4\operatorname{cosec}^2\theta$

$\dfrac{dL}{d\theta} = 0 \Rightarrow 3\sec^2\theta = 4\operatorname{cosec}^2\theta$

$\dfrac{3}{\cos^2\theta} = \dfrac{4}{\sin^2\theta}$

$\tan^2\theta = \dfrac{4}{3}$

$\tan\theta = \dfrac{2}{\sqrt{3}} \left(\text{rej.} -\dfrac{2}{\sqrt{3}} \because 0 < \theta < \dfrac{\pi}{2}\right)$

Stationary value of $OP + OQ = 7 + 3\left(\dfrac{2}{\sqrt{3}}\right) + 4\left(\dfrac{\sqrt{3}}{2}\right)$

$= 7 + 4\sqrt{3}$

Many students were unable to find the x-coordinate of point P.

For students who found the expression for $OP + OQ$, they were unable to differentiate the expression.

Students should know the following:

(1) $\dfrac{d}{d\theta}(\tan\theta) = \sec^2\theta$

(2) $\dfrac{d}{d\theta}(\cot\theta) = -\operatorname{cosec}^2\theta$