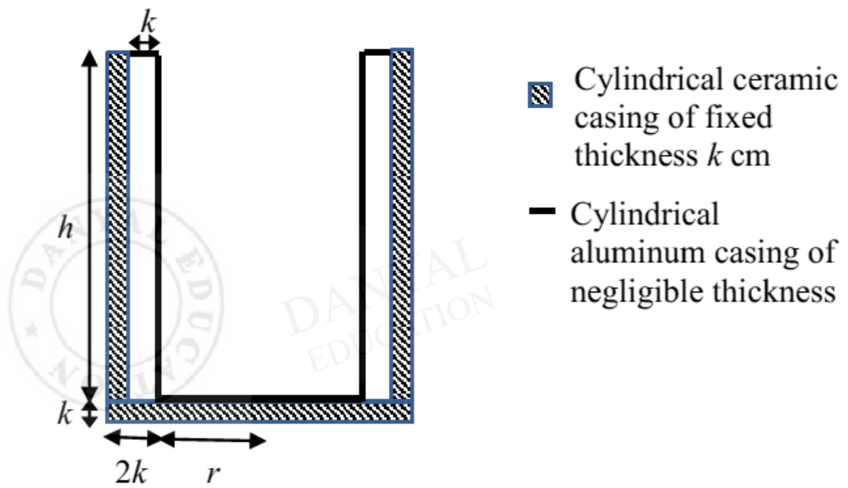
A Level H2 Math

Differentiation and its Applications Test 10

Q1

A company intends to manufacture a cylindrical double-walled ceramic vacuum flask which can hold a fixed V cm^3 of liquid when filled to the brim. The cylindrical vacuum flask is made up of an inner cylindrical aluminum casing (of negligible thickness) with height h cm and radius r cm and an outer cylindrical ceramic casing of fixed thickness k cm. There is a fixed k cm gap between the sides of the inner casing and outer casing where air has been removed to form a vacuum. The diagram below shows the view of the vacuum flask if it is dissected vertically through the centre.



Let the volume of the outer ceramic casing be C cm^3.

(i) Show that the volume of the ceramic casing can be expressed as

$$C = k\left(\frac{2V}{r} + \frac{3kV}{r^2} + \pi\left(r+2k\right)^2\right).$$ [4]

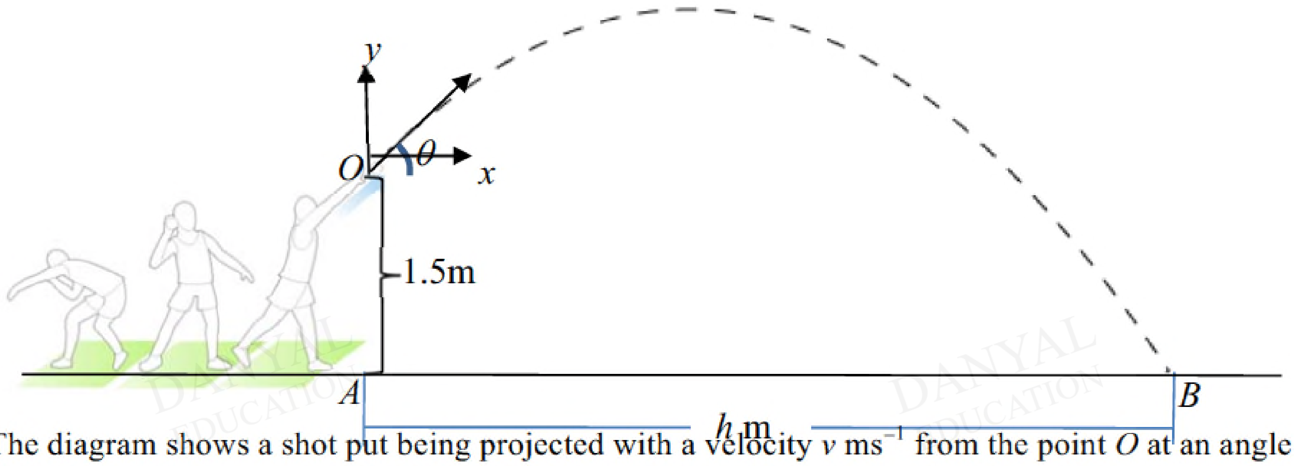
(ii) Let r_1 be the value of r which gives the minimum value of C. Show that r_1 satisfies the equation $\pi r^4 + 2\pi k r^3 - rV - 3kV = 0$. [3]

For the rest of the question, it is given that $k = \frac{1}{4}$ and $V = 250$.

(iii) Find the minimum volume of the ceramic casing, proving that it is a minimum. [3]

(iv) Sketch the graph showing the volume of the ceramic casing as the radius of the aluminum casing varies. [2]

Q2



The diagram shows a shot put being projected with a velocity v ms^{-1} from the point O at an angle θ made with the horizontal. The point O is 1.5m above the point A on the ground. The x-y plane is taken to be the plane that contains the trajectory of this projectile motion with x-axis parallel to the horizontal and O being the origin. The equation of the trajectory of this projectile motion is known to be

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta},$$

where g ms^{-2} is the acceleration due to gravity.
The constant g is taken to be 10 and the distance between A and B is denoted by h m. Given that $v = 10$, show that h satisfies the equation

$$h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0$$ [3]

As θ varies, h varies. Show that stationary value of h occurs when θ satisfies the following equation

$$3\tan^2 2\theta - 20\sin 2\theta\tan 2\theta - 20\cos 2\theta - 20 = 0.$$ [5]

Hence find the stationary value of h. [2]

Q3

A hot air balloon rises vertically upwards from the ground as the balloon operator intermittently fires and turns off the burner. At time t minutes, the balloon ascends at a rate inversely proportional to $t+\lambda$, where λ is a positive constant. At the same time, due to atmospheric factors, the balloon descends at a rate of 2 km per minute. It is also known that initially the rate of change of the height of the balloon is 1 km per minute.

(i) Find a differential equation expressing the relation between H and t, where H km is the height of the hot air balloon above ground at time t minutes. Hence solve the differential equation and find H in terms of t and λ. [7]

Using $\lambda = 15$,

(ii) Find the maximum height of the balloon above ground in exact form. [3]

(iii) Find the total vertical distance travelled by the balloon when $t = 8$. [3]

(iv) Can we claim that the rate of change of the height of the balloon above the ground is decreasing? Explain your answer. [2]

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Answers

Differentiation and its Applications Test 10

Q1

(i)

$$V = \pi h r^2$$

$$h = \frac{V}{\pi r^2} \quad \text{--------} \quad (*)$$

$$C = \pi(h+k)(r+2k)^2 - \pi h(r+k)^2$$

$$= \pi\left(h\left((r+2k)^2 - (r+k)^2\right) + k(r+2k)^2\right)$$

$$= \pi\left(h\left((r^2+4rk+4k^2) - (r^2+2rk+k^2)\right) + k(r+2k)^2\right)$$

$$= \pi\left(\frac{V}{\pi r^2}(2rk+3k^2) + k(r+2k)^2\right) \qquad (\text{from } (*))$$

$$= k\left(V\frac{(2r+3k)}{r^2} + \pi(r+2k)^2\right)$$

$$= k\left(\frac{2V}{r} + \frac{3kV}{r^2} + \pi(r+2k)^2\right) \qquad (\text{shown})$$

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(ii) $$\frac{dC}{dr} = k\left(\frac{-2V}{r^2} - \frac{6kV}{r^3} + 2\pi(r+2k)\right)$$

When $\frac{dC}{dr} = 0$,

$$k\left(\frac{-2V}{r^2} - \frac{6kV}{r^3} + 2\pi(r+2k)\right) = 0$$

$$-Vr - 3kV + \pi r^3(r+2k) = 0$$

$$\pi r^4 + 2k\pi r^3 - Vr - 3kV = 0 \qquad (\text{Shown})$$

(iii) From GC,

$r_1 = 4.3736 \quad (\text{since } r > 0)$

r	r_1^-	r_1	r_1^+
$\frac{dC}{dr}$			

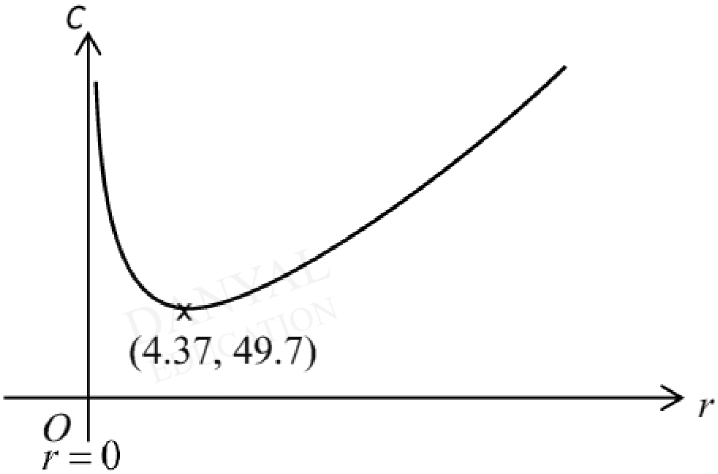
$$\frac{d^2C}{dr^2} = 5.33 > 0 \Rightarrow C \text{ is a minimum}$$

$C = 49.7 \quad (3 \text{ s.f.})$

Minimum volume of ceramic casing is 49.7 cm^3.

Note: cannot use part (iv) graph to solve this part.

(iv)



Q2

$$y = x\tan\theta - \frac{10x^2}{2(10)^2\cos^2\theta}$$

$$\Rightarrow \quad y = x\tan\theta - \frac{x^2}{20\cos^2\theta}$$

When $y = -1.5, \quad x = h$

$$\therefore \qquad -1.5 = h\tan\theta - \frac{h^2}{20\cos^2\theta}$$

$$\Rightarrow \qquad -30\cos^2\theta = 20h\tan\theta\cos^2\theta - h^2$$

$$\Rightarrow \qquad h^2 - 20h\sin\theta\cos\theta - 30\cos^2\theta = 0$$

$$\Rightarrow \qquad h^2 - 10h\sin 2\theta - 15(1+\cos 2\theta) = 0$$

$$\Rightarrow \qquad h^2 - 10h\sin 2\theta - 15\cos 2\theta - 15 = 0 \quad \text{--- (*)}$$

(Shown)

Differentiate both sides w.r.t. θ, we have

$$2h\frac{dh}{d\theta} - 10\frac{dh}{d\theta}\sin 2\theta - 20h\cos 2\theta + 30\sin 2\theta = 0$$

At stationary value, $\frac{dh}{d\theta} = 0$.

$$\therefore \qquad -20h\cos 2\theta + 30\sin 2\theta = 0$$

$$\Rightarrow \qquad h = \frac{30\sin 2\theta}{20\cos 2\theta} = \frac{3}{2}\tan 2\theta$$

Sub into (*), we have

$$\left(\frac{3}{2}\tan 2\theta\right)^2 - 10\left(\frac{3}{2}\tan 2\theta\right)\sin 2\theta - 15\cos 2\theta - 15 = 0$$

$$\Rightarrow \quad \frac{9}{4}\tan^2 2\theta - 15\tan 2\theta\sin 2\theta - 15\cos 2\theta - 15 = 0$$

$$\Rightarrow \quad 3\tan^2 2\theta - 20\sin 2\theta\tan 2\theta - 20\cos 2\theta - 20 = 0$$

(Shown)

Using GC, $\theta = 0.71999$ (5 sig fig)

Therefore, max $h = \frac{3}{2}\tan 2(0.71999) = 11.4$ (3 sig fig)

Marker's comments

This question is poorly attempted in general.

(i) Students who fail to get credit for this part do not realise that $y = -1.5$ when $x = h$. There were also signs which indicate that students have difficulty applying the double-angle formula.

(ii) One common mistake made by students is to differentiate with respect to *h*. This is a conceptual error which indicates a poor understanding of derivatives. Many students on the other hand chose to make *h* the subject before differentiating, failing to realise that implicit differentiation would get the job done much easily. There was also a recurring problem of product rule when differentiating $10h\sin 2\theta$, with the erroneous result of $10\frac{\mathrm{d}h}{\mathrm{d}\theta}(-2\cos 2\theta)$.

(iii) The equation can be easily solved using the GC, though there were many attempts to solve it algebraically. Students using the GC in degree mode would fail to obtain any credit for this part.

Q3

(i) Rate of increase in height = $\dfrac{k}{t+\lambda}$ where k is a positive constant Rate of decrease in height = 2 Therefore, $\dfrac{\mathrm{d}H}{\mathrm{d}t}=\dfrac{k}{t+\lambda}-2$ Since $\dfrac{\mathrm{d}H}{\mathrm{d}t}=1$ when $t=0$, we have $1=\dfrac{k}{0+\lambda}-2$ $\Rightarrow 1=\dfrac{k-2\lambda}{\lambda}\;\therefore k=3\lambda$ Hence, $\dfrac{\mathrm{d}H}{\mathrm{d}t}=\dfrac{3\lambda}{t+\lambda}-2$ (Do not combine into one single fraction!) Integrating wrt t: $H=\int\left(\dfrac{3\lambda}{t+\lambda}-2\right)\mathrm{d}t=3\lambda\ln\lvert t+\lambda\rvert-2t+C$ Since $t+\lambda>0$, we have $H=3\lambda\ln(t+\lambda)-2t+C$ When $t=0$, $H=0$: $0=3\lambda\ln(\lambda)+C\;\therefore C=-3\lambda\ln\lambda$ $H=3\lambda\ln(t+\lambda)-2t-3\lambda\ln\lambda$ $\therefore H=3\lambda\ln\left(\dfrac{t}{\lambda}+1\right)-2t$ (ii) Using $\lambda=15$, at maximum height $\dfrac{\mathrm{d}H}{\mathrm{d}t}=\dfrac{45}{t+15}-2=0$ $\therefore t=7.5$ $\therefore H=45\ln\left(\dfrac{7.5}{15}+1\right)-2(7.5)=15\left(3\ln\dfrac{3}{2}-1\right)$	At least a few students in each class wrote rate of increase in height = $\dfrac{t+\lambda}{k}$, $k(t+\lambda)$ or $\dfrac{1}{k(t+\lambda)}$. Obviously they do not know the meaning of inversely proportional. Some leave their final answer as the general solution without finding the value of C.

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(iii) When $t = 8$,

$$H = 45\ln\left(\frac{8}{15}+1\right)-2(8) = 45\ln\frac{23}{15}-16$$

Total vertical distance travelled
= Vertical distance travelled from $t = 0$ to $t = 7.5$ +
Vertical distance travelled from $t = 7.5$ to $t = 8$

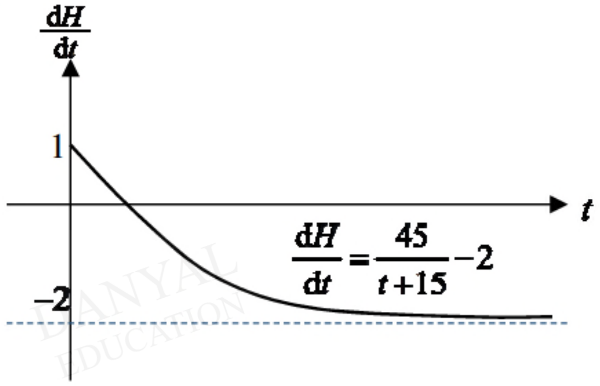
$$= 15\left(3\ln\frac{3}{2}-1\right)+\left[15\left(3\ln\frac{3}{2}-1\right)-45\ln\frac{23}{15}+16\right]$$

= 3.26 km (correct to 3 s.f.)

(iv) $\frac{\mathrm{d}^2H}{\mathrm{d}t^2} = \frac{-45}{(t+15)^2} < 0$ for all real values of t, $t \geq 0$

i.e. the rate of change of the height of the balloon above ground is decreasing.

Or from the graph of $\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{45}{t+15}-2$, we see that $\frac{\mathrm{d}H}{\mathrm{d}t}$ decreases as t increases.



Marker's comments

For part (i):

- For those who managed to get the correct DE, most are able to solve the DE using direct integration. Students lose marks if modulus is not included after integration or no reason is provided for dropping modulus.
- Students need to know that they are to find H in terms of t and λ, which means they need to find C by interpreting from the question that at $t = 0$, $H = 0$.

For part (ii):

Part (ii) was well done with only a few students not knowing how to approach the question. A few students did not read the question carefully and did not leave their answer for maximum height in exact form.

For part (iii):

Part (iii) was badly done. Many students did not realise that maximum height is reached at $t = 7.5$ (from (ii)), which means that H will decrease after 7.5 mins. Many students simply find H when $t = 8$. Some went to integrate $\frac{\mathrm{d}H}{\mathrm{d}t}$ from $t = 0$ to $t = 8$ which is incorrect.

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For part (iv):

This part was also badly done. Many students conclude that as $t \to \infty$, $\frac{\mathrm{d}H}{\mathrm{d}t} \to -2$ and thus rate of change of height is decreasing, having the misconception that H decreases then rate of change of the height of the balloon is also decreasing. Some explain by drawing the graph of H instead of $\frac{\mathrm{d}H}{\mathrm{d}t}$.

Students need to know that if we want to show that H decreases with t, we need to show that $\frac{\mathrm{d}H}{\mathrm{d}t} < 0$. Similarly, if we want to show that rate of change of H, i.e. $\frac{\mathrm{d}H}{\mathrm{d}t}$ is decreasing, we need to show that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2H}{\mathrm{d}t^2} < 0$ or draw the graph of $\frac{\mathrm{d}H}{\mathrm{d}t}$ and show that it is decreasing with increasing t.