A Level H2 Math

Differentiation and its Applications Test 1

Q1

The curve C and the line L have equations $y = x^2$ and $y = \frac{1}{2}x - 2$ respectively.

(i) The point A on C and the point B on L are such that they have the same x-coordinate. Find the coordinates of A and B that gives the shortest distance AB. [3]

(ii) The point P on C and the point Q on L are such that they have the same y-coordinate. Find the coordinates of P and Q that gives the shortest distance PQ. [3]

(iii) Find the exact area of the polygon formed by joining the points found in **(i)** and **(ii)**. [2]

(iv) A variable point on the curve C with coordinates $\left(s, s^2\right)$ starts from the origin O and moves along the curve with s increasing at a rate of 2 units/s. Find the rate of change of the area bounded by the curve, the y-axis and the line $y = s^2$, at the instant when $s = \sqrt{2}$. [4]

Q2

A curve C has parametric equations

$$x = \cos t$$

$$y = \frac{1}{2}\sin 2t$$

where $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$.

(i) Find the equation of the normal to C at the point P with parameter p. [2]

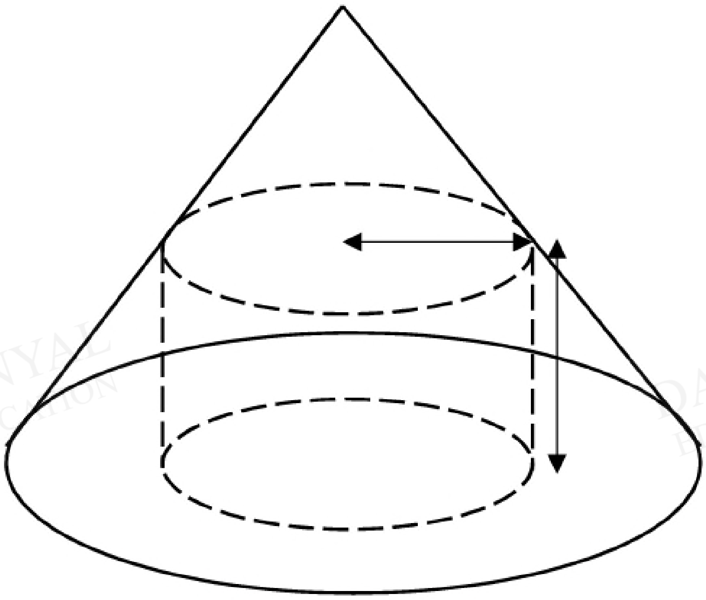
The normal to C at the point when $t = \frac{2\pi}{3}$ cuts the curve again. Find the coordinates of the point of intersection. [2]

(ii) Sketch C, clearly labelling the coordinates of the points where the curve crosses the x- and y- axes. [1]

(iii) Find the cartesian equation of C. [2]

The region bounded by C is rotated through π radians about the x-axis. Find the exact volume of the solid formed. [3]

Q3



A metal cylinder of radius r cm and height h cm is inscribed in a circular cone paperweight of base radius 4 cm and height 6 cm (see diagram).

It is determined that the volume of the cylinder, V cm^3, should be as large as possible to provide weight to the paperweight. Show that

$$V = \frac{4\pi}{9}\left(36h - 12h^2 + h^3\right).$$ [2]

Hence find the exact maximum value of V. [5]

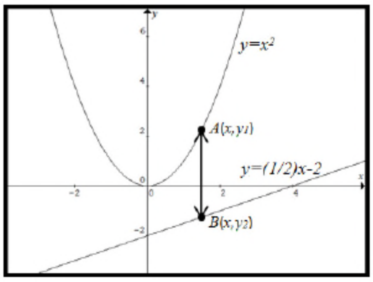
The metal cylinder is known to expand under heat. An experiment shows that the height of the cylinder is increasing at a rate of 0.04 cm s^{-1} at an instant when $h = 1.5$. Find the rate of change of V at this instant. [2]

Answers

Differentiation and its Applications Test 1

Q1

(i) Let V be the distance AB.



$V = y_1 - y_2$

$= x^2 - \left(\frac{1}{2}x - 2\right)$

$= x^2 - \frac{1}{2}x + 2$

$\frac{dV}{dx} = 2x - \frac{1}{2}$

when $\frac{dV}{dx} = 0,\ x = \frac{1}{4}$

$\frac{d^2V}{dx^2} = 2 > 0 \Rightarrow$ min. value when $x = \frac{1}{4}$

when $x = \frac{1}{4}$,

$y = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

$y = \frac{1}{2}\left(\frac{1}{4}\right) - 2 = -\frac{15}{8}$

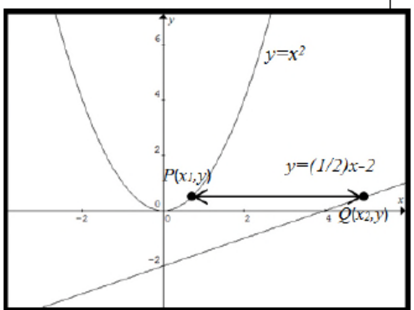
$\therefore$ coords on C (Pt A): $\left(\frac{1}{4}, \frac{1}{16}\right)$ & coords on L (Pt B): $\left(\frac{1}{4}, -\frac{15}{8}\right)$.

For many, distance was not even considered, instead look at gradients of L and C. Those who used distance, some were penalised for not checking nature of stationary value. Many students made slips in simple calculations such as

$2x - \frac{1}{2} \Rightarrow x = 1,$

$y^{-\frac{1}{2}} = \frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$ etc.

(ii) Let H be the distance PQ.



$H = x_2 - x_1 = 2(y+2) - \sqrt{y}$

$\frac{dH}{dy} = 2 - \frac{1}{2}y^{-\frac{1}{2}}$

when $\frac{dH}{dy} = 0$,

$2 - \frac{1}{2}y^{-\frac{1}{2}} = 0 \Rightarrow 2 = \frac{1}{2}y^{-\frac{1}{2}}$

$\Rightarrow y = 4^{-2} = \frac{1}{16}$

$\frac{d^2H}{dy^2} = \frac{1}{4}y^{-\frac{3}{2}}$

$\Rightarrow$ when $y = \frac{1}{16}, \frac{d^2H}{dy^2} = \frac{1}{4}\left(\frac{1}{16}\right)^{-\frac{3}{2}} = 16 > 0$

$\Rightarrow$ min. value when $y = \frac{1}{16}$

when $y = \dfrac{1}{16}$,

$x = \sqrt{\dfrac{1}{16}} = \dfrac{1}{4}$

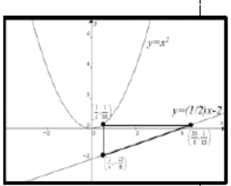
$x = 2\left(\dfrac{1}{16}\right) + 2 = \dfrac{33}{8}$

$\therefore$ coords on C (Pt P): $\left(\dfrac{1}{4}, \dfrac{1}{16}\right)$ & coords on L (Pt Q): $\left(\dfrac{33}{8}, \dfrac{1}{16}\right)$.

(iii) Area of polygon = Area of triangle

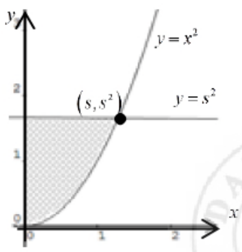
Minimum distance $AB = \dfrac{1}{16} - \left(-\dfrac{15}{8}\right) = \dfrac{31}{16}$

Minimum distance $PQ = \dfrac{33}{8} - \left(\dfrac{1}{4}\right) = \dfrac{31}{8}$



$\therefore$ Area of polygon $= \dfrac{1}{2} \times \dfrac{31}{16} \times \dfrac{31}{8} = \dfrac{961}{256}$ sq units

(iv)



$\dfrac{\mathrm{d}s}{\mathrm{d}t} = 2$

Method 1:

$$\text{Area} = A = \int_0^{s^2} x \ \mathrm{d}y = \int_0^{s^2} \sqrt{y} \ \mathrm{d}y = \left[\frac{y^{\frac{3}{2}}}{3/2}\right]_0^{s^2} = \frac{2}{3}s^3$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}s} \times \frac{\mathrm{d}s}{\mathrm{d}t} = 2s^2 \times 2 = 4s^2$$

$\therefore$ when $s = \sqrt{2}$, $\dfrac{\mathrm{d}A}{\mathrm{d}t} = (4)\left(\sqrt{2}\right)^2 = 8$ units²/s

Well answered except those who treated area bounded as a constant instead of a variable, hence were clueless as to how to get $\dfrac{\mathrm{d}A}{\mathrm{d}s}$.
When finding area, confused by the variable point, many students did not use definite integral.

Method 2:

Area = A

= Area of rectangle – Area bounded by curve, x-axis and $x = s$

$$= s \times s^2 - \int_0^s y \ \mathrm{d}x = s^3 - \int_0^s x^2 \ \mathrm{d}x = s^3 - \left[\frac{x^3}{3}\right]_0^s = \frac{2}{3}s^3$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}s} \times \frac{\mathrm{d}s}{\mathrm{d}t} = 2s^2 \times 2 = 4s^2$$

$\therefore$ when $s = \sqrt{2}, \Rightarrow \dfrac{\mathrm{d}A}{\mathrm{d}t} = 4\left(\sqrt{2}\right)^2 = 8$ units²/s

Q2

(i)	$x=\cos t$ $y=\frac{1}{2}\sin 2t$ $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\cos 2t}{-\sin t}$ $\left.\frac{dy}{dx}\right\vert_{t=p}=\frac{\cos 2p}{-\sin p}\Rightarrow$ gradient of normal $=\frac{\sin p}{\cos 2p}$ $\Rightarrow$ equation of normal at $\left(\cos p,\frac{1}{2}\sin 2p\right)$: $y-\frac{1}{2}\sin 2p=\frac{\sin p}{\cos 2p}(x-\cos p)$ $y=\frac{\sin p}{\cos 2p}x+\frac{1}{2}(\sin 2p-\tan 2p)$	Generally students were able to write down the eqn of normal at point with parameter p. However, some wrote $\frac{dy}{dx}=\frac{dy}{dp}\times\frac{dp}{dx}$. Although no mark is deducted here, students should realize that p in most cases is a constant (though not specified by question) and $\frac{dy}{dp}=0$. A minority wrote the eqn of normal as $y-\frac{1}{2}\sin 2p=\frac{\sin t}{\cos 2t}(x-\cos p)$ without putting $t=p$. Many careless mistakes in evaluating the cosine and sine values when $t=\frac{2\pi}{3}$, resulting in wrong eqns of

$\Rightarrow$ equation of normal at $t=\dfrac{2\pi}{3}$:

$$y=\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}x+\frac{1}{2}\left(-\frac{\sqrt{3}}{2}-\sqrt{3}\right)\Rightarrow y=-\sqrt{3}x-\frac{1}{4}\left(3\sqrt{3}\right)....(1)$$

To find point of intersection of normal and C (when the normal cuts C again),

Substitute $x=\cos t$ and $y=\dfrac{1}{2}\sin 2t$ into (1):

$$\frac{1}{2}\sin 2t=-\sqrt{3}\left(\cos t\right)-\frac{1}{4}\left(3\sqrt{3}\right)$$

$$\frac{1}{2}\sin 2t+\sqrt{3}\left(\cos t\right)+\frac{1}{4}\left(3\sqrt{3}\right)=0$$

From GC,

$t=2.094395$ (corresponds to $t=\dfrac{2\pi}{3}$)

or $t=3.495928$

$\Rightarrow$ point normal meets C again:

$$\left(\cos\left(3.495928\right),\frac{1}{2}\sin\left(2\left(3.495928\right)\right)\right)=\left(-0.938,0.325\right)$$

normal, such as

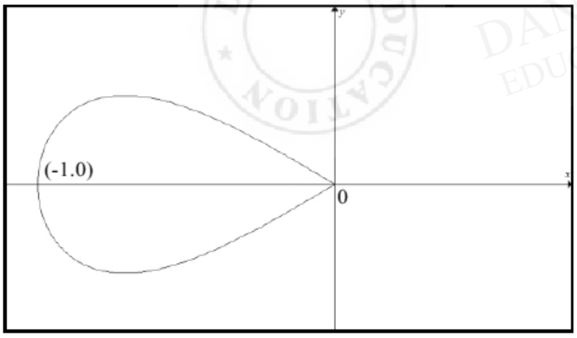
$y=-\sqrt{3}x-\dfrac{\sqrt{3}}{4}$,

$y=\sqrt{3}x-\dfrac{3\sqrt{3}}{4}$ etc

Many did not understand that the question is asking for point of intersection **between the curve and the normal at** $t=\dfrac{2\pi}{3}$ and simply sub $t=\dfrac{2\pi}{3}$ to find the point.

Those who correctly sub $x=\cos t$ and $y=\dfrac{1}{2}\sin 2t$ into (1) often did not use GC to solve the eqn, and simply stopped at this step.

(ii)



Many did not note the range of values of t and sketched 2 loops.

A number of students did not give the coordinates of the x-intercept.

(iii) Method 1:

$x=\cos t\Rightarrow x^2=\cos^2 t$

$y=\dfrac{1}{2}\sin 2t\Rightarrow y=\sin t\cos t$

$\Rightarrow y^2=\sin^2 t\cos^2 t=\left(1-\cos^2 t\right)\cos^2 t=\left(1-x^2\right)x^2$

$\therefore$ Cartesian equation: $y^2=\left(1-x^2\right)x^2$

Method 2:

$x=\cos t\Rightarrow \cos t=\dfrac{x}{1},\sin t=\dfrac{\pm\sqrt{1-x^2}}{1}\quad\left(\because \dfrac{\pi}{2}\le t\le\dfrac{3\pi}{2}\right)$

$y=\dfrac{1}{2}\sin 2t\Rightarrow y=\sin t\cos t=\pm\sqrt{1-x^2}\left(x\right)$

$\therefore$ Cartesian equation: $y=\pm x\sqrt{1-x^2}$

Many simply wrote the eqn as $y=\sin 2(\cos^{-1}x)$ and did not go on to simplify.

Those who used method 2 often omitted the negative sign.

Method 3: $x=\cos t \Rightarrow x^2=\cos^2 t \Rightarrow \cos 2t = 2\cos^2 t - 1 = 2x^2-1$ $y=\frac{1}{2}\sin 2t \Rightarrow \sin 2t = 2y$ Using $\sin^2 2t+\cos^2 2t=1$, $(2y)^2+(2x^2-1)^2=1$ $\therefore$ Cartesian equation: $4y^2+(2x^2-1)^2=1$	
Method 1: $\int_{-1}^{0}\pi y^2\,dx$ $=\pi\int_{-1}^{0}(1-x^2)x^2\,dx$ $=\pi\int_{-1}^{0}x^2-x^4\,dx=\pi\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_{-1}^{0}=\frac{2}{15}\pi$ units^3 Method 2 (not advised): $x=\cos t \Rightarrow \frac{dx}{dt}=-\sin t$ when $x=0,\ t=\frac{\pi}{2},\frac{3\pi}{2}$ (can use either) when $x=-1,\ t=\pi$ $\int_{-1}^{0}\pi y^2\,dx$ $=\pi\int_{\pi}^{\frac{3\pi}{2}}\left(\frac{1}{2}\sin 2t\right)^2(-\sin t)\,dt$ $=-\pi\int_{\pi}^{\frac{3\pi}{2}}(\sin t\cos t)^2(\sin t)\,dt$ $=-\pi\int_{\pi}^{\frac{3\pi}{2}}\sin^2 t\cos^2 t(\sin t)\,dt$ $=-\pi\int_{\pi}^{\frac{3\pi}{2}}(1-\cos^2 t)\cos^2 t(\sin t)\,dt$ $=-\pi\int_{\pi}^{\frac{3\pi}{2}}(\cos^2 t-\cos^4 t)(\sin t)\,dt$ $=-\pi\left(-\int_{\pi}^{\frac{3\pi}{2}}(\cos t)^2(-\sin t)\,dt+\int_{\pi}^{\frac{3\pi}{2}}(\cos t)^4(-\sin t)\,dt\right)$ $=-\pi\left(-\left[\frac{(\cos t)^3}{3}\right]_{\pi}^{\frac{3\pi}{2}}+\left[\frac{(\cos t)^5}{5}\right]_{\pi}^{\frac{3\pi}{2}}\right)$ $=-\pi\left(-0-\frac{1}{3}+0+\frac{1}{5}\right)=\frac{2}{15}\pi$ units^3	Many did not realize that method 1 is the desired method and were stucked with method 2 as they did not know how to integrate the integrand. For method 2, common mistakes include wrong limits, or writing volume as $2\int_{-1}^{0}\pi y^2\,dx$.

Q3

Using similar triangles: $\frac{r}{4}=\frac{6-h}{6}$

$r=\frac{2}{3}(6-h)$

$$\begin{aligned}V&=\pi r^2 h\\&=\pi\left(\frac{2}{3}(6-h)\right)^2 h\\&=\frac{4\pi}{9}\left(36-12h+h^2\right)h\\&=\frac{4\pi}{9}\left(36h-12h^2+h^3\right)\quad\text{(shown)}\end{aligned}$$

For maximum V, $\frac{\mathrm{d}V}{\mathrm{d}h}=0$:

$\frac{4\pi}{9}\left(36-24h+3h^2\right)=0$

Using GC: $h=2$ or $h=6$ (Rejected as $h=6$ is height of cone)

Method 1 (1st derivative sign test)

h	2^-	2	2^+
Sign of $\frac{\mathrm{dV}}{\mathrm{dh}}$	+	0	-
slope	/	—	\

Thus, maximum volume $V=\frac{128\pi}{9}$ when $h=2$ cm.

Method 2 (2nd derivative test)

$\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=\frac{4\pi}{9}(-24+6h)$

When $h=2$: $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=-\frac{16\pi}{3}<0$

Thus, maximum volume $V=\frac{128\pi}{9}$.

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}&=\frac{\mathrm{d}V}{\mathrm{d}h}.\frac{\mathrm{d}h}{\mathrm{d}t}\\&=\frac{4\pi}{9}\left(36-24(1.5)+3(1.5)^2\right)(0.04)\\&=0.12\pi\quad\mathrm{cm^3s^{-1}}\qquad\text{(Accept: } 0.377\ \mathrm{cm^3s^{-1}})\end{aligned}$$