A Level H2 Math

Differential Equations Test 5

Q1

A population of 15 foxes has been introduced into a national park. A zoologist believes that the population of foxes, x, at time t years, can be modelled by the Gompertz equation given by:

$$\frac{dx}{dt} = cx\ln\left(\frac{40}{x}\right)$$

where c is a constant.

(i) Using the substitution $u = \ln\left(\frac{40}{x}\right)$, show that the differential equation can be written as $\frac{du}{dt} = -cu$. **[2]**

(ii) Hence find u in terms of t and show that $x = 40e^{-Be^{-ct}}$, where B is a constant. **[5]**

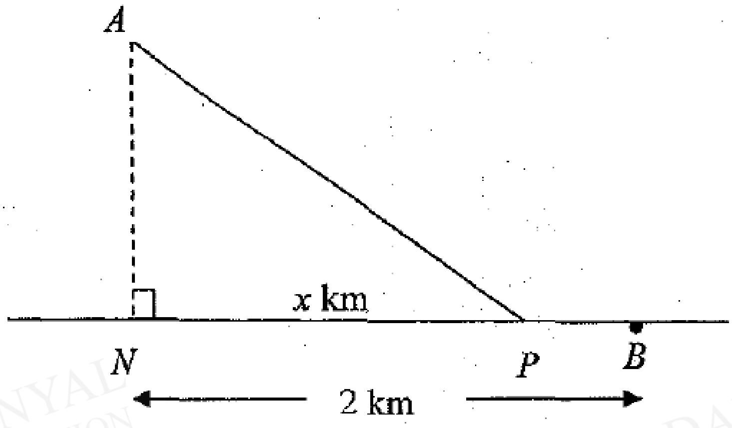
After 3 years, the population of foxes is estimated to be 20.

(iii) Find the values of B and c. **[3]**

(iv) Find the population of foxes in the long run. **[1]**

(v) Hence, sketch the graph showing the population of foxes over time. **[2]**

Q2



Alvin is at the point A on a floating platform in the sea. He wants to reach point B located on a straight stretch of beach. N is the point on the beach nearest to A and $NB = 2$ km. Alvin swims at a constant speed in a straight line from A to P and then runs at a constant speed from P to B, where P is a point on the straight stretch of beach from N to B. $NP = x$ km and T minutes is the time taken for Alvin to complete the journey.

T and x satisfy the differential equation

$$\frac{dT}{dx} = \frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5.$$

(i) Solve the differential equation. [3]

(ii) Given that the minimum time taken for Alvin to complete this journey is 30 minutes, find T in terms of x. [3]

(iii) Using your answer in part **(ii)**, find the longest time taken by Alvin to complete the journey. [2]

Q3

The growth of an organism in a controlled environment is monitored and the growth rate of the organism is proportional to $(N-x)x$, where x is the population (in thousands) of the organism at time t and N is a constant such that $x<N$. The initial population of the organism is $\frac{1}{3}N$.

(i) Find x in terms of t and determine the population of the organism in the long run, giving your answer in terms of N. [6]

Another model is proposed for the growth of the organism, which assumes the growth rate is purely a function of time and is modelled by the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{-9t}{\left(4+9t^2\right)^2}$. It predicts that the population of the organism will also eventually stabilise.

(ii) Show that under this model, $x=\frac{1}{12}\tan^{-1}\left(\frac{3t}{2}\right)+\frac{N}{3}$.

Hence state the population of the organism in the long run, giving your answer in terms of N. [6]

Answers

Differential Equations Test 5

Q1

(i)

$$\frac{dx}{dt} = cx\ln\left(\frac{40}{x}\right)$$

$$u = \ln\left(\frac{40}{x}\right)$$
$$= \ln(40) - \ln(x)$$

$$\frac{du}{dx} = -\frac{1}{x}$$

$$\frac{du}{dt} = \frac{du}{dx} \times \frac{dx}{dt}$$
$$= \left(-\frac{1}{x}\right)cx\ln\left(\frac{40}{x}\right)$$
$$= -cu$$

(ii)

$$\frac{du}{dt} = -cu$$

$$\int \frac{1}{u}\,du = -\int c\ dt$$
$$\ln|u| = -ct + d$$
$$|u| = e^{-ct+d}$$
$$u = \pm e^{d}e^{-ct}$$
$$= Be^{-ct},\ B = \pm e^{d}$$

Replace u with $\ln\left(\frac{40}{x}\right)$

$$\ln\left(\frac{40}{x}\right) = Be^{-ct}$$

$$\frac{40}{x} = e^{Be^{-ct}}$$

$$x = \frac{40}{e^{Be^{-ct}}}$$

$$x = 40e^{-Be^{-ct}}$$

(iii)

When $t = 0,\ x = 15,$

$15 = 40e^{-B}$

$$e^{-B} = \frac{3}{8}$$

$$B = \ln\left(\frac{8}{3}\right) = 0.98083 = 0.981$$

When $t = 3,\ x = 20$

$$20 = 40e^{-Be^{-3c}}$$

$$e^{-Be^{-3c}} = \frac{1}{2}$$

$$-Be^{-3c} = \ln\frac{1}{2}$$

$$\ln\left(\frac{3}{8}\right)\left(e^{-3c}\right) = \ln\left(\frac{1}{2}\right)$$

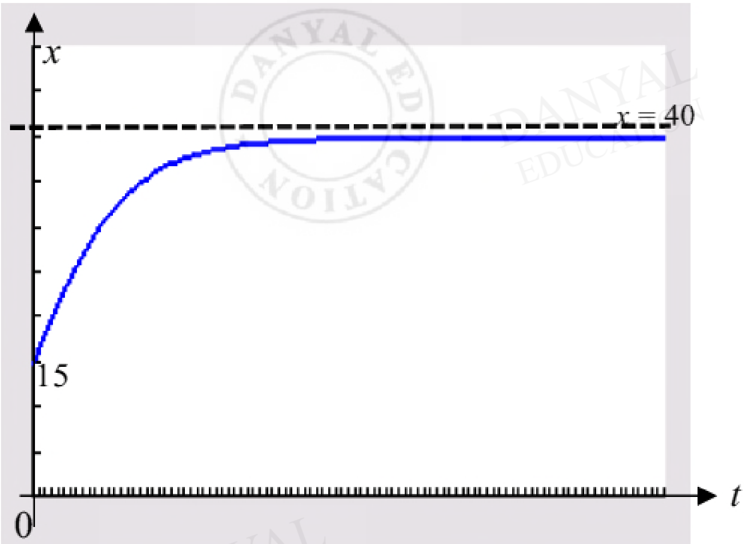
$$c = -\frac{1}{3}\ln\left(\frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{3}{8}\right)}\right) = 0.11572 = 0.116$$

$$x = 40e^{-0.981e^{-0.116t}}$$

(iv)

The population of foxes in the long run is 40.

(v)



Q2

5i

$$\frac{\mathrm{d}T}{\mathrm{d}x}=\frac{5\sqrt{5}x}{\sqrt{x^2+4}}-5$$

$$T=\int\left(\frac{5\sqrt{5}x}{\sqrt{x^2+4}}-5\right)\mathrm{d}x$$

$$=\frac{5\sqrt{5}}{2}\int 2x\left(x^2+4\right)^{-\frac{1}{2}}\,\mathrm{d}x-\int 5\,\mathrm{d}x$$

$$=\frac{5\sqrt{5}}{2}\frac{\left(x^2+4\right)^{\frac{1}{2}}}{\frac{1}{2}}-5x+C$$

$$=5\sqrt{5}\left(x^2+4\right)^{\frac{1}{2}}-5x+C \qquad ---(1)$$

ii

When $t=30$, $\frac{\mathrm{d}T}{\mathrm{d}x}=0$: $\frac{5\sqrt{5}x}{\sqrt{x^2+4}}=5\Rightarrow\sqrt{5}x=\sqrt{x^2+4}$

$5x^2=x^2+4\Rightarrow x=\pm1$

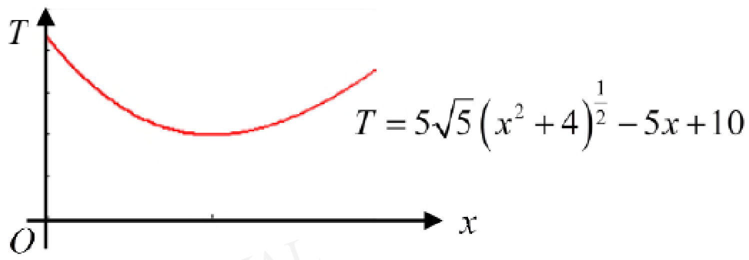
Since $x>0$, $x=1$

Substitute $x=1$ and $T=30$ into equation (1)

$30=5\sqrt{5}\left(1+4\right)^{\frac{1}{2}}-5+C\Rightarrow C=10$

$T=5\sqrt{5}\left(x^2+4\right)^{\frac{1}{2}}-5x+10$

iii



When $x=0$, $T=32.361$. When $x=2$, $T=31.623$

Longest time taken by Alvin is 32.4 mins.

Q3

(i)

$$\frac{dx}{dt}=k(N-x)x$$

$$\frac{1}{(N-x)x}\frac{dx}{dt}=k$$

$$\int\frac{1}{(N-x)x}\,dx=\int k\,dt$$

$$\frac{1}{N}\int\frac{1}{N-x}+\frac{1}{x}\,dx=\int k\,dt$$

$$\frac{1}{N}\left(-\ln|N-x|+\ln|x|\right)=kt+C$$

Remember to include the modulus sign whenever the integral involves ln.

$$\frac{1}{N}\ln\left|\frac{x}{N-x}\right|=kt+C$$

$$\ln\left|\frac{x}{N-x}\right|=Nkt+NC$$

$$\left|\frac{x}{N-x}\right|=e^{Nkt+NC}$$

$$\frac{x}{N-x}=Ae^{Nkt}\quad\text{where } A=\pm e^{NC}$$

When $t=0$, $x=\frac{1}{3}N$,

$$A=\frac{1}{2}$$

$$\frac{x}{N-x}=\frac{1}{2}e^{Nkt}$$

$$2x=(N-x)e^{Nkt}$$

$$x\left(2+e^{Nkt}\right)=Ne^{Nkt}$$

$$x=\frac{Ne^{Nkt}}{2+e^{Nkt}}\qquad\text{equivalently, } x=\frac{N}{2e^{-Nkt}+1}$$

<u>Alternative method</u> (not recommended):

$$\frac{1}{(N-x)x}\frac{\mathrm{d}x}{\mathrm{d}t}=k$$

$$-\int\frac{1}{x^2-Nx}\,\mathrm{d}x=\int k\,\mathrm{d}t$$

$$-\int\frac{1}{\left(x-\frac{N}{2}\right)^2-\left(\frac{N}{2}\right)^2}\,\mathrm{d}x=\int k\,\mathrm{d}t$$

$$-\left(\frac{1}{2\left(\frac{N}{2}\right)}\ln\left|\frac{\left(x-\frac{N}{2}\right)-\frac{N}{2}}{\left(x-\frac{N}{2}\right)+\frac{N}{2}}\right|\right)=kt+C$$

$$-\left(\frac{1}{N}\ln\left|\frac{x-N}{x}\right|\right)=kt+C$$

$$\frac{1}{N}\ln\left|\frac{x}{N-x}\right|=kt+C$$

$$\ln\left(\frac{x}{N-x}\right)=Nkt+NC \quad \text{since } 0<x<N$$

We can remove the modulus sign by taking into account the range of values the variable x can take

When $t=0$, $x=\frac{1}{3}N$, $\ln\frac{1}{2}=NC\Rightarrow C=-\frac{1}{N}\ln 2$

$$\ln\left(\frac{x}{N-x}\right)=Nkt-\ln 2$$

$$\ln\left(\frac{2x}{N-x}\right)=Nkt$$

$$\frac{2x}{N-x}=\mathrm{e}^{Nkt}$$

$$x\left(2+\mathrm{e}^{Nkt}\right)=N\mathrm{e}^{Nkt}$$

$$x=\frac{N\mathrm{e}^{Nkt}}{2+\mathrm{e}^{Nkt}} \qquad \text{equivalently, } x=\frac{N}{2\mathrm{e}^{-Nkt}+1}$$

As $t\to\infty$, $x=\frac{N}{2\mathrm{e}^{-Nkt}+1}\to N$

(ii)

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}=\frac{-9t}{\left(4+9t^2\right)^2}$$

$$\int\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\,\mathrm{d}t=\int\frac{-9t}{\left(4+9t^2\right)^2}\,\mathrm{d}t$$

$$=-\frac{1}{2}\int\frac{18t}{\left(4+9t^2\right)^2}\,\mathrm{d}t$$

Integral $\int\frac{18t}{\left(4+9t^2\right)^2}\,\mathrm{d}t$ is of the form $\int \mathrm{f}'(t)\left[\mathrm{f}(t)\right]^n\,\mathrm{d}t$ where $n=-2$.

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{2}\left(\frac{1}{4+9t^2}\right)+A$$

$$\int\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t=\frac{1}{2}\int\frac{1}{4+9t^2}\,\mathrm{d}t+\int A\,\mathrm{d}t$$

$$=\frac{1}{18}\int\frac{1}{\left(\frac{2}{3}\right)^2+t^2}\,\mathrm{d}t+\int A\,\mathrm{d}t$$

$$=\frac{1}{18}\cdot\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{t}{\frac{2}{3}}\right)+At+B$$

Some students wrongly express the integral as $\frac{1}{2}\int\frac{1}{4+9t^2}+A\ \mathrm{d}t$.

Note that $\int\frac{1}{2}\left(\frac{1}{4+9t^2}\right)+A\ \mathrm{d}t=\frac{1}{2}\int\frac{1}{4+9t^2}+\underline{2}A\ \mathrm{d}t$.

Thus it is recommended to express as two separate integrals as shown in the solution.

To find $\int\frac{1}{4+9t^2}\ \mathrm{d}t$, it is necessary to have the coefficient of t^2 to be 1 before applying formula in MF26. Thus s

$$\int\frac{1}{4+9t^2}\ \mathrm{d}t=\frac{1}{9}\int\frac{1}{\left(\frac{2}{3}\right)^2+t^2}\ \mathrm{d}t=\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{t}{\frac{2}{3}}\right)=\frac{3}{2}\tan^{-1}\left(\frac{3t}{2}\right)$$

and not $\int\frac{1}{4+9t^2}\ \mathrm{d}t=\int\frac{1}{2^2+(3t)^2}\ \mathrm{d}t=\frac{1}{2}\tan^{-1}\left(\frac{3t}{2}\right)$.

$$x=\frac{1}{12}\tan^{-1}\left(\frac{3t}{2}\right)+At+B$$

Since the population stabilises in the long run, as $t\to\infty$, $x\to$ finite value, $A=0$

When $t=0$, $x=\frac{1}{3}N$, $B=\frac{N}{3}$

Hence $x=\frac{1}{12}\tan^{-1}\left(\frac{3t}{2}\right)+\frac{N}{3}$

When $t\to\infty$, $\tan^{-1}\left(\frac{3t}{2}\right)\to\frac{\pi}{2}$

Hence $x\to\frac{\pi}{24}+\frac{N}{3}$.