A Level H2 Math

Differential Equations Test 4

Q1

(a) **(i)** Show that $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta - \frac{1}{3}\sin^3\theta\right) = \cos^3\theta$. [1]

(ii) Find the solution to the differential equation $\operatorname{cosec} x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\cos^2 x$ in the form $y = \mathrm{f}(x)$, given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3} + \frac{2}{\pi}$ when $x = 0$. [4]

(b) Show, by means of the substitution $v = x^2 y$, that the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y + 4x^2 y = 0$$

can be reduced to the form

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -4vx .$$

Hence find y in terms of x, given that $y = \frac{1}{3}$ when $x = -3$. [6]

Q2

A population of a certain organism grows from an initial size of 5. After 5 days, the size of the population is 20, and after t days, the size of the population is M. The rate of growth of the population is modelled as being proportional to $\left(100^2 - M^2\right)$.

(i) Write down a differential equation modelling the population growth and find M in terms of t. [6]

(ii) Find the size of the population after 15 days, giving your answer correct to the nearest whole number. [2]

(iii) Find the least number of days required for the population to exceed 80. [2]

Q3

In a model of forest fire investigation, the proportion of the total area of the forest which has been destroyed is denoted by x. The destruction rate of the fire is defined to be the rate of change of x with respect to the time t, in hours, measured from the instant the fire is first noticed. A particular forest fire is initially noticed when 20% of the total area of the forest is destroyed.

(a) One model of forest fire investigation shows that the destruction rate is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{10}x(1-x).$$

(i) Express the solution of the differential equation in the form $x = \mathrm{f}(t)$ and sketch the part of the curve for $t \geq 0$. [6]

(ii) Find the exact time when the destruction rate is at its maximum. [2]

(iii) Explain briefly why this model cannot be used to estimate how long the forest has been burning when it is first noticed. [1]

(b) A second model for the investigation of forest fire is suggested and given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{5\pi\left[1+\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)^2\right]}.$$

Determine how long the forest has been burning when the fire is first noticed. [3]

<u>Answers</u>

<u>Differential Equations Test 4</u>

Q1

(a) (i) $$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)$$

$$=\cos\theta-\sin^2\theta\cos\theta$$

$$=\cos\theta\left(1-\sin^2\theta\right)$$

$$=\cos\theta\left(\cos^2\theta\right)=\cos^3\theta$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\sin x\cos^2 x$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=(-\sin x)(\cos x)^2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(\cos x)^3}{3}+C$$

$$=\frac{1}{3}\left(\cos x\cdot\cos^2 x\right)+C$$

$$=\frac{1}{3}\left(\cos x\cdot\left(1-\sin^2 x\right)\right)+C$$

$$=\frac{1}{3}\left(\cos x-\cos x\cdot\sin^2 x\right)+C$$

$y=\frac{1}{3}\left(\sin x-\frac{\sin^3 x}{3}\right)+Cx+D$

When $x=0$ and $y=0$, $D=0$

When $x=0$ and $\frac{dy}{dx}=\frac{1}{3}+\frac{2}{\pi}$, $C=\frac{2}{\pi}$

$y=\frac{1}{3}\left(\sin x-\frac{\sin^3 x}{3}\right)+\frac{2}{\pi}x$

(b) $v=x^2y$ --------------------(1)

$\frac{dv}{dx}=2xy+x^2\frac{dy}{dx}$ -------- (2)

$x\frac{dy}{dx}+2y+4x^2y=0$ ----- (3)

(3) × x, $x^2\frac{dy}{dx}+2xy+4x^2y(x)=0$ -------- (4)

$\frac{dv}{dx}+4x\left(x^2y\right)=0$

$\frac{dv}{dx}+4vx=0$

$\frac{dv}{dx}=-4vx$ (Shown)

$\frac{dv}{dx}=-4vx$

$\int\frac{1}{v}\,dv=-4\int x\,dx$

$\ln|v|=-2x^2+c$

$v=\pm e^{-2x^2+c}$

$v=Ae^{-2x^2}$, where $A=\pm e^c$

$x^2y=Ae^{-2x^2}$

Given that $y=\frac{1}{3}$ when $x=-3$,

$(-3)^2\left(\frac{1}{3}\right)=Ae^{-18}$

$A=3e^{18}$

$y=\frac{3e^{18-2x^2}}{x^2}$

Q2

(i) $\frac{dM}{dt} = k\left(100^2 - M^2\right)$, $k > 0$

Since $\frac{dM}{dt} > 0$ and $M > 0, \Rightarrow \left(100^2 - M^2\right) > 0$ and $0 < M < 100$

$$\int \frac{1}{\left(100^2 - M^2\right)} dM = \int k \, dt$$

$$\frac{1}{200}\ln\left(\frac{100+M}{100-M}\right) = kt + C$$

$$\ln\left(\frac{100+M}{100-M}\right) = 200kt + C'$$

$$\frac{100+M}{100-M} = Ae^{200kt} \text{ , where } A = e^{C'}$$

When $t = 0$, $M = 5 \Rightarrow A = \frac{105}{95} = \frac{21}{19}$

When $t = 5$, $M = 20 \Rightarrow \frac{3}{2} = \frac{21}{19}e^{1000k}$

$$e^{1000k} = \frac{19}{14} \quad or \quad 200k = \frac{1}{5}\ln\left(\frac{19}{14}\right)$$

Thus $\frac{100+M}{100-M} = \frac{21}{19}\left(e^{1000k}\right)^{\frac{t}{5}} = \frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}$

$$100 + M = \frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}(100 - M)$$

$$M\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}} + 1\right] = 100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}} - 1\right]$$

$$M = \frac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}} - 1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}} + 1} \quad OR \quad \frac{100\left[21\left(\frac{19}{14}\right)^{\frac{t}{5}} - 19\right]}{21\left(\frac{19}{14}\right)^{\frac{t}{5}} + 19} \quad OR \quad \frac{100\left[\left(\frac{19}{14}\right)^{\frac{t}{5}} - \frac{19}{21}\right]}{\left(\frac{19}{14}\right)^{\frac{t}{5}} + \frac{19}{21}}$$

(ii)

When $t=15$, $M=\dfrac{100\left[\dfrac{21}{19}\left(\dfrac{19}{14}\right)^{3}-1\right]}{\dfrac{21}{19}\left(\dfrac{19}{14}\right)^{3}+1}=46.847$

$M\approx 47$ (nearest whole number)

(iii)

Method 1: Graphical Method

Sketch the graphs of M=f(t) and M=80

From the graph, when $t>34.336397$, $M>80$

Least number of days required is 35.

Method 2: Use GC table

When $t=34$, $M=79.627<80$

When $t=35$, $M=80.718>80$

When $t=36$, $M=81.756>80$

$\Rightarrow t\geq 35$

Thus least number of days required is 35.

Method 3:

$$\frac{100\left[\dfrac{21}{19}\left(\dfrac{19}{14}\right)^{\frac{t}{5}}-1\right]}{\dfrac{21}{19}\left(\dfrac{19}{14}\right)^{\frac{t}{5}}+1}>80$$

$$\frac{5}{4}\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]>\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1$$

$$\frac{1}{4}\cdot\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}>\frac{9}{4}$$

$$\left(\frac{19}{14}\right)^{\frac{t}{5}}>\frac{57}{7}$$

$$t>\frac{5\ln\left(\dfrac{57}{7}\right)}{\ln\left(\dfrac{19}{14}\right)}=34.336397$$

Least number of days required is 35.

Q3

(a)(i) $$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{10}x(1-x)$$

$$\Rightarrow \int\left(\frac{1}{x}+\frac{1}{1-x}\right)\mathrm{d}x=\int\frac{1}{10}\,\mathrm{d}t$$

$$\Rightarrow \ln\left|\frac{x}{1-x}\right|=\frac{1}{10}t+C\ ,$$

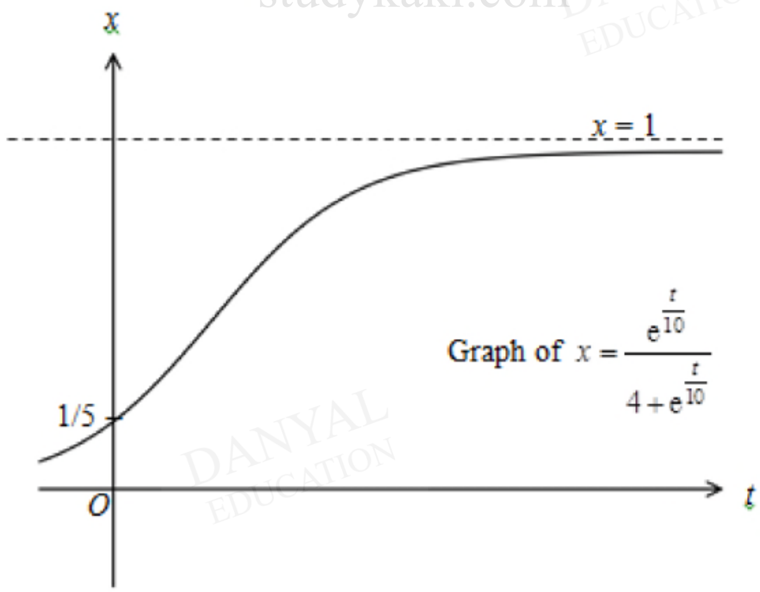
where C is an arbitrary constant

$$\Rightarrow \frac{x}{1-x}=Ae^{\frac{t}{10}}\ ,\ \text{where } A=\pm e^{C}$$

$$\Rightarrow x=\frac{Ae^{\frac{t}{10}}}{1+Ae^{\frac{t}{10}}}$$

When $t=0$, $x=\frac{1}{5}$. That is, $\frac{1}{5}=\frac{A}{1+A}\Rightarrow A=\frac{1}{4}$.

Hence, $x=\dfrac{\frac{1}{4}e^{\frac{t}{10}}}{1+\frac{1}{4}e^{\frac{t}{10}}}=\dfrac{e^{\frac{t}{10}}}{4+e^{\frac{t}{10}}}=1-\dfrac{4}{4+e^{\frac{t}{10}}}$.



(a)(ii) Since $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{10}x(1-x)$ is a quadratic expression, destruction rate is at its maximum when $x=\frac{0+1}{2}=\frac{1}{2}$

Or when $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)=\frac{1}{10}-\frac{1}{5}x=0$ i.e., $x=\frac{1}{2}$.

Therefore, $\dfrac{\frac{1}{2}}{1-\frac{1}{2}}=\dfrac{1}{4}e^{\frac{t}{10}}$.

$\Rightarrow \quad e^{\frac{t}{10}}=4 \quad \Rightarrow \quad t=10\ln 4$

(a)(iii) From $x=\dfrac{e^{\frac{t}{10}}}{4+e^{\frac{t}{10}}}$, $\quad x=0 \Rightarrow e^{\frac{t}{10}}=0$.

Graph of $x=\dfrac{e^{\frac{t}{10}}}{4+e^{\frac{t}{10}}}$:

Since $e^{\frac{t}{10}}>0$ for all real t, there is no value of t for $x=0$.

OR Note that $x=1-\dfrac{4}{4+e^{\frac{t}{10}}}$.

Since $0<\dfrac{4}{4+e^{\frac{t}{10}}}<1$, we will have $0<x<1$.

Hence, $x>0$ for all real values of t, and there is no value of t for $x=0$.

OR As $t\to-\infty$, $e^{\frac{t}{10}}\to 0^+$, $x=1-\dfrac{4}{4+e^{\frac{t}{10}}}\to 0^+$.

Hence, $x=0$ is a horizontal asymptote and there are no values of t giving $x=0$.

(b) $\dfrac{dx}{dt}=\dfrac{1}{5\pi\left[1+\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)^2\right]}$

$$x=\frac{1}{5\pi}\int\frac{1}{1+\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)^2}\,dt$$

$$=\frac{10}{5\pi}\int\frac{\frac{1}{10}}{1+\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)^2}\,dt$$

$$=\frac{2}{\pi}\tan^{-1}\left(\frac{t}{10}+\tan\frac{\pi}{10}\right)+C$$

When $t=0$, $x=\dfrac{1}{5}$. Hence

$$\frac{1}{5}=\frac{2}{\pi}\tan^{-1}\left(\tan\frac{\pi}{10}\right)+C \Rightarrow C=0$$

That is, $x=\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{10}+\tan\dfrac{\pi}{10}\right)$.

From G.C., when $x=0$, $t=-3.25$ (3 s.f.)

Hence, the forest have been burning for 3.25 hours when it is first noticed.