**A Level H2 Math**

**Differential Equations Test 3**

Q1

To determine whether the amount of preservatives in a particular brand of bread meets the safety limit of preservatives present, the Food Regulatory Authority (FRA) conducted a test to examine the growth of fungus on a piece of bread over time after its expiry date. The piece of bread has a surface area of $100 \text{ cm}^2$. The staff from FRA estimate the amount of fungus grown and the rate at which it is growing by finding the area of the piece of bread the fungus covers over time. They believe that the area, $A \text{ cm}^2$, of fungus present $t$ days after the expiry date is such that the rate at which the area is increasing is proportional to the product of the area of the piece of bread covered by the fungus and the area of the bread not covered by the fungus. It is known that the initial area of fungus is $20 \text{ cm}^2$ and that the area of fungus is $40 \text{ cm}^2$ five days after the expiry date.

**(i)** Write down a differential equation expressing the relation between $A$ and $t$. [1]

**(ii)** Find the value of $t$ at which 50% of the piece of bread is covered by fungus, giving your answer correct to 2 decimal places. [6]

**(iii)** Given that this particular brand of bread just meets the safety limit of the amount of preservatives present when the test is concluded 2 weeks after the expiry date, find the range of values of $A$ for any piece of bread of this brand to be deemed safe for human consumption in terms of the amount of preservatives present, giving your answer correct to 2 decimal places. [2]

**(iv)** Write the solution of the differential equation in the form $A = \mathrm{f}(t)$ and sketch this curve. [3]

Q2

Show that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3xy}{1-3x^2} - x + 1 = 0$$

may be reduced by means of the substitution $y = u\sqrt{1-3x^2}$ to

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x-1}{\sqrt{1-3x^2}}.$$

Hence find the general solution for $y$ in terms of $x$. [5]

Q3

At the intensive care unit of a hospital, patients of a particular condition receive a certain treatment drug through an intravenous drip at a constant rate of 30mg per hour. Due to the limited capacity for absorption by the body, the drug is lost from a patient's body at a rate proportional to $x$ , where $x$ is the amount of drug (in mg) present in the body at time $t$ (in hours). It is assumed that there is no presence of the drug in any patient prior to admission to the hospital.

(i) Form a differential equation involving $x$ and $t$ and show that $x=\frac{30}{k}\left(1-\mathrm{e}^{-kt}\right)$ where $k$ is a positive constant. [4]

(ii) If there is more than 1000mg of drug present in a patient's body, it is considered an overdose. Suppose the drug continues to be administered, determine the range of values of $k$ such that a patient will have an overdose. [2]

For a particular patient, $k=\frac{1}{50}$.

(iii) Find the time required for the amount of the drug present in the patient's body to be 200mg. [3]

**Answers**

**Differential Equations Test 3**

Q1

**(i)** $\frac{\mathrm{d}A}{\mathrm{d}t} = kA(100-A)$

**(ii)**
$$\int \frac{1}{A(100-A)}\,\mathrm{d}A = \int k\ dt$$

By partial fractions,
$$\frac{1}{A(100-A)} = \frac{1}{100A} + \frac{1}{100(100-A)}$$

$$\therefore \frac{1}{100}\int\left(\frac{1}{A} + \frac{1}{100-A}\right)\mathrm{d}A = kt + c$$

$$\frac{1}{100}\left(\ln|A| - \ln|100-A|\right) = kt + c \quad (\because A > 0 \text{ and } 100 - A > 0)$$

$$\frac{1}{100}\left[\ln A - \ln(100-A)\right] = kt + c$$

$$\ln\frac{A}{100-A} = 100(kt+c)$$

$$\frac{A}{100-A} = \mathrm{e}^{100(kt+c)} = \mathrm{e}^{100kt}\mathrm{e}^{100c} = D\mathrm{e}^{k_1 t}$$

where $k_1 = 100k$ and $D = \mathrm{e}^{100c}$.

When $t = 0, A = 20$,

$$\frac{20}{100-20} = D$$

$$D = \frac{1}{4}$$

When $t = 5, A = 40$,

$$\begin{aligned}
\frac{40}{100-40} &= \frac{1}{4}e^{5k_1} \\
\frac{1}{4}e^{5k_1} &= \frac{2}{3} \\
e^{5k_1} &= \frac{8}{3} \\
5k_1 &= \ln\frac{8}{3} \\
k_1 &= \frac{1}{5}\ln\frac{8}{3} \\
\therefore \frac{A}{100-A} &= \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}
\end{aligned}$$

When $A = 0.5\times100 = 50$,

$$\begin{aligned}
\frac{50}{100-50} &= \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t} \\
1 &= \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t} \\
e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t} &= 4 \\
\left(\frac{1}{5}\ln\frac{8}{3}\right)t &= \ln 4 \\
t &= \frac{\ln 4}{\frac{1}{5}\ln\frac{8}{3}} = 7.07 \text{ (2 dp)}
\end{aligned}$$

The required time is 7.07 days.

**(iii)** When $t = 14$ (days),

$$\frac{A}{100-A} = \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)(14)}$$

<u>Method 1 Solve algebraically</u>

$$\frac{A}{100-A} = 3.8963 \text{ (5 sf)}$$
$$A = (100-A)(3.8963)$$
$$= 389.63 - 3.8963A$$
$$4.8963A = 389.63$$
$$A = 79.58 \text{ (2 dp)}$$

For the bread to be deemed safe for human consumption in terms of the amount of preservatives present, $\underline{\underline{79.58 \le A \le 100}}$

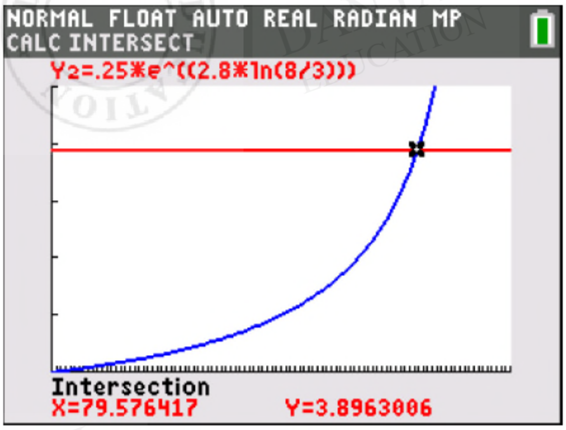
<u>Method 2 Use GC to plot graphs</u>

Use GC to plot $y = \dfrac{A}{100-A}$ and $y = \dfrac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)(14)}$ $(\approx 3.8963)$

Look for the point of intersection (adjust window).

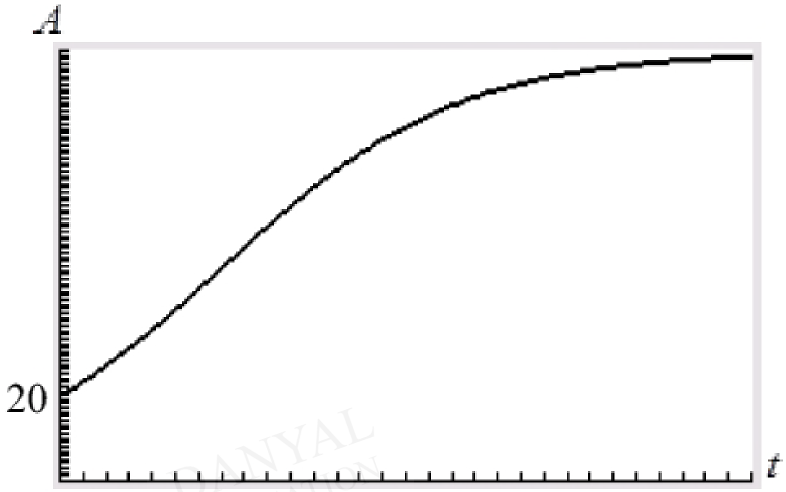
$$A = 79.58 \text{ (2 dp)}$$

For the bread to be deemed safe for human consumption in terms of the amount of preservatives present, $\underline{\underline{79.58 \le A \le 100}}$




**(iv)**

$$\frac{A}{100-A} = \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$
$$A = \frac{1}{4}e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}(100-A)$$
$$4A = e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}(100-A) = 100e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t} - Ae^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$
$$\left[4+e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}\right]A = 100e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}$$
$$A = \underline{\underline{\frac{100e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}}{4+e^{\left(\frac{1}{5}\ln\frac{8}{3}\right)t}}}} \text{ or } \underline{\underline{\frac{100e^{0.196t}}{4+e^{0.196t}}}}$$


A
20
t

Q2

Let $y = u\sqrt{1-3x^2}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx}\sqrt{1-3x^2} + u\left(\frac{1}{2}\right)\frac{-6x}{\sqrt{1-3x^2}}$$

DE : $\dfrac{dy}{dx} + \dfrac{3xy}{1-3x^2} - x + 1 = 0$

$$\Rightarrow \quad \frac{du}{dx}\sqrt{1-3x^2} - \frac{-3xu}{\sqrt{1-3x^2}} + \frac{3x}{1-3x^2}\left(u\sqrt{1-3x^2}\right) - x + 1 = 0$$

$$\Rightarrow \quad \frac{du}{dx}\sqrt{1-3x^2} - \frac{3xu}{\sqrt{1-3x^2}} + \frac{3xu}{\sqrt{1-3x^2}} = x - 1$$

$$\Rightarrow \quad \frac{du}{dx}\sqrt{1-3x^2} = x - 1$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{x}{\sqrt{1-3x^2}} - \frac{1}{\sqrt{1-3x^2}}$$

$$\Rightarrow \quad u = -\frac{1}{6}\int \frac{-6x}{\sqrt{1-3x^2}}dx - \int \frac{1}{\sqrt{1-3x^2}}dx$$

$$\Rightarrow \quad \frac{y}{\sqrt{1-3x^2}} = -\frac{1}{6}\left[2\sqrt{1-3x^2}\right] - \frac{\sin^{-1}\left(\sqrt{3}x\right)}{\sqrt{3}} + C$$

$$\Rightarrow \quad y = -\frac{1}{3}\left(1-3x^2\right) - \frac{\sqrt{1-3x^2}}{\sqrt{3}}\sin^{-1}\left(\sqrt{3}x\right) + C\sqrt{1-3x^2}$$

Q3

**1(i)**

$$\frac{dx}{dt} = 30 - kx, \qquad k > 0$$

$$\Rightarrow \int \frac{1}{30 - kx}\,dx = \int dt$$

$$\Rightarrow -\frac{1}{k}\ln|30 - kx| = t + C$$

$$\Rightarrow \ln|30 - kx| = -kt - kC$$

$$\Rightarrow |30 - kx| = e^{-kt-kC}$$

$$\Rightarrow 30 - kx = Ae^{-kt}, \qquad \text{where } A = \pm e^{-kC}$$

$$\Rightarrow x = \frac{1}{k}\left(30 - Ae^{-kt}\right)$$

At $t = 0$, $x = 0 \Rightarrow 0 = \frac{1}{k}\left(30 - Ae^{0}\right) \Rightarrow A = 30$

$$\Rightarrow x = \frac{1}{k}\left(30 - 30e^{-kt}\right) = \frac{30}{k}\left(1 - e^{-kt}\right)$$

**1(ii)**

For patient to have overdose,

$$x = \frac{30}{k}\left(1 - e^{-kt}\right) > 1000$$

Since for $t > 0$, $0 < e^{-kt} < 1$, so $0 < 1 - e^{-kt} < 1$

$$\frac{30}{k} > \frac{30}{k}\left(1 - e^{-kt}\right) > 1000$$

$$0 < k < \frac{30}{1000} = 0.03$$

**(iii)**

At $x = 200$, $\quad 200 = 30(50)\left(1 - e^{-\frac{t}{50}}\right)$

$$1 - e^{-\frac{t}{50}} = \frac{2}{15}$$

$$t = 50\ln\left(\frac{15}{13}\right)$$

Using GC, $t = 7.16$h or 7h 9min