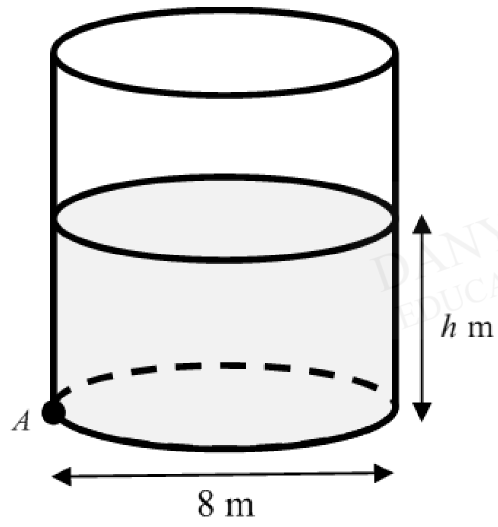
A Level H2 Math

Differential Equations Test 2

Q1



The figure above shows a cylindrical water tank with base diameter 8 metres. Water is flowing into the tank at a constant rate of $0.36\pi\,\text{m}^3$/min. At time t minutes, the depth of water in the tank is h metres. However, the tank has a small hole at point A located at the bottom of the tank. Water is leaking from point A at a rate of $0.8\pi h$ m^3/min.

(i) Show that the depth, h metres, of the water in the tank at time, t minutes satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{400}(9-20h).$$ [3]

(ii) Given that $h=0.4$ when $t=0$, find the particular solution of the above differential equation in the form $h=\mathrm{f}(t)$. [6]

(iii) Explain whether the tank will be emptied. [1]

(iv) Sketch the part of the curve with the equation found in part **(ii)**, which is relevant in this context. [2]

Q2

(a) By using the substitution $u = \frac{y}{x}$, show that the differential equation

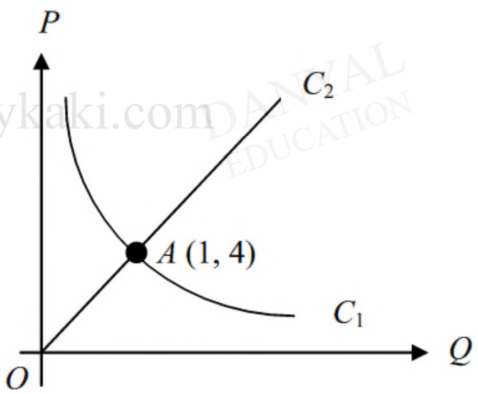
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 + xy + x^2}{x^2}, \text{ where } x > 0,$$

can be reduced to $\frac{1}{u^2+1}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$. Hence, find y in terms of x. [5]

(b) In the diagram below, the curve C_1 and the line C_2 illustrate the relationship between price (P dollars per kg) and quantity (Q tonnes) for consumers and producers respectively.

The curve C_1 shows the quantity of rice that consumers will buy at each price level while the line C_2 shows the quantity of rice that producers will produce at each price level. C_1 and C_2 intersect at point A, which has the coordinates (1, 4).

The quantity of rice that consumers will buy is inversely proportional to the price of the rice. The quantity of rice that producers will produce is directly proportional to the price.



(i) Interpret the coordinates of A in the context of the question. [1]

(ii) Solve for the equations of C_1 and C_2, expressing Q in terms of P. [2]

Shortage occurs when the quantity of rice consumers will buy exceeds the quantity of rice producers will produce. It is known that the rate of increase of P after time t months is directly proportional to the quantity of rice in shortage.

(iii) Given that the initial price is \$3 and that after 1 month, the price is \$3.65, find P in terms of t and sketch this solution curve, showing the long-term behaviour of P.

[7]

Suggest a reason why producers might use $P = aQ + b$, where $a, b \in \mathbb{R}^+$, instead of C_2 to model the relationship between price and quantity of rice produced. [1]

Q3

Food energy taken in by a man goes partly to maintain the healthy functioning of his body and partly to increase body mass. The total food energy intake of the man per day is assumed to be a constant denoted by I (in joules). The food energy required to maintain the healthy functioning of his body is proportional to his body mass M (in kg). The increase of M with respect to time t (in days) is proportional to the energy not used by his body. If the man does not eat for one day, his body mass will be reduced by 1%.

(i) Show that I, M and t are related by the following differential equation:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{I - aM}{100a}, \text{ where } a \text{ is a constant.}$$

State an assumption for this model to be valid. [3]

(ii) Find the total food energy intake per day, I, of the man in terms of a and M if he wants to maintain a constant body mass. [1]

It is given that the man's initial mass is 100kg.

(iii) Solve the differential equation in part **(i)**, giving M in terms of I, a and t. [3]

(iv) Sketch the graph of M against t for the case where $I > 100a$. Interpret the shape of the graph with regard to the man's food energy intake. [3]

(v) If the man's total food energy intake per day is $50a$, find the time taken in days for the man to reduce his body mass from 100kg to 90kg. [2]

Answers

Differential Equations Test 2

Q1

(i)

$$V = \pi\left(4^2\right)h$$

$$\frac{dV}{dh} = 16\pi$$

$$\frac{dV}{dt} = \frac{dV_{in}}{dt} - \frac{dV_{out}}{dt}$$
$$= 0.36\pi - 0.8\pi h$$

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$0.36\pi - 0.8\pi h = 16\pi\cdot\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{0.36\pi - 0.8\pi h}{16\pi}$$

$$\frac{dh}{dt} = \frac{1}{400}(9-20h)\text{ (shown)}$$

This part is usually well done.

Some candidates introduced t, representing time, in an attempt to establish an equation of h in terms of t. Followed by wrong differentiation of h with respect to t .This approach earn no mark.

(ii)

$$\frac{dh}{dt} = \frac{1}{400}(9-20h)$$

$$\int \frac{1}{9-20h}\,dh = \frac{1}{400}\int 1\,dt$$

$$-\frac{1}{20}\int \frac{-20}{9-20h}\,dh = \frac{1}{400}\int 1\,dt$$

$$-\frac{1}{20}\ln|9-20h| = \frac{1}{400}(t+A)$$

$$\ln|9-20h| = -\frac{1}{20}(t+A)$$

$$|9-20h| = e^{-\frac{1}{20}(t+A)}$$

$$9-20h = \pm e^{-\frac{1}{20}(t+A)}$$

$$9-20h = \pm e^{-\frac{1}{20}t}\cdot e^{-\frac{1}{20}A}$$

$$9-20h = Be^{-\frac{1}{20}t}\text{ where } B = \pm e^{-\frac{1}{20}A}$$

When $t=0$, $h=0.4$,

$$9-20(0.4) = Be^{-\frac{1}{20}(0)}$$

$$B = 1$$

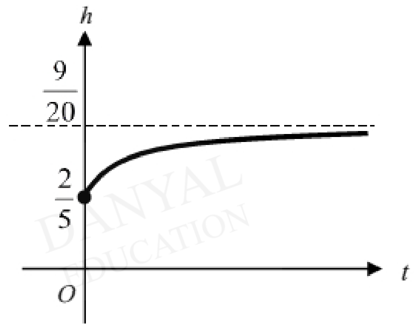
$$\therefore\ 9-20h = e^{-\frac{1}{20}t}$$

$$20h = 9 - e^{-\frac{1}{20}t}$$

$$h = \frac{1}{20}\left(9 - e^{-\frac{1}{20}t}\right)$$

Careless mistakes in writing the numbers are unusually frequent in this part and resulted in marks loss.

Generally well done.

(iii)	If the tap is on indefinitely, the tank will not be empty. In the long run, there will be $\frac{9}{20}$ m of water in the tank.	No marks awarded to candidates who attempted to explain in words without clear reference to the mathematical equation obtained earlier.
(iv)	$h=\frac{1}{20}\left(9-\mathrm{e}^{-\frac{1}{20}t}\right)$ 	Many candidates often overlooked the presence of a horizontal asymptote. Lacks proper labelling of axes or asymptote.

Q2

(a)

$u=\frac{y}{x}\Rightarrow y=ux$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2+xy+x^2}{x^2}=\left(\frac{y}{x}\right)^2+\frac{y}{x}+1$ ---(1)

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}x+u$ -- (2)

Sub (2) into (1):

$\frac{\mathrm{d}u}{\mathrm{d}x}x+u=u^2+u+1\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}x=u^2+1$

$\frac{1}{u^2+1}\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$\int\frac{1}{u^2+1}\,\mathrm{d}u=\int\frac{1}{x}\,\mathrm{d}x$

$\tan^{-1}u=\ln|x|+c$, where c is an arbitrary constant.

$\tan^{-1}u=\ln x+c$ (since $x>0$)

$u=\tan(\ln x+c)$

$y=x\tan(\ln x+c)$

(b)(i)

Point A shows that at 4 dollars per kg, 1 tonne of rice is produced and all of it is bought by the consumers.

This is the equilibrium point where the price is 4 dollars per kg and the quantity produced/consumed is 1 tonne.

(b)(ii)

$C_1:Q=\frac{k_1}{P}$

$C_2:Q=k_2P$

When $Q=1,P=4$,

$k_1=4,k_2=\frac{1}{4}$.

$C_1:Q=\frac{4}{P};C_2:Q=\frac{P}{4}$

Hence, $C_1 : Q = \frac{4}{P}; C_2 : Q = \frac{p}{4}$.

(b)(iii)

$$\frac{dP}{dt} = k_3\left(\frac{4}{P} - \frac{P}{4}\right)$$

$$\frac{dP}{dt} = k_3\left(\frac{16-P^2}{4P}\right)$$

$$\int \frac{4P}{16-P^2}\,dP = \int k_3\,dt$$

$$-2\int \frac{-2P}{16-P^2}\,dP = \int k_3\,dt$$

$$-2\ln\left|16-P^2\right| = k_3 t + c$$

$$\ln\left|16-P^2\right| = \frac{-k_3}{2}t + \frac{-c}{2}$$

$$\left|16-P^2\right| = e^{\frac{-k_3}{2}t+\frac{-c}{2}}$$

$$16-P^2 = \left(\pm e^{\frac{-c}{2}}\right)e^{\frac{-k_3}{2}t}$$

$$16-P^2 = Ae^{Bt}, A = \pm e^{\frac{-c}{2}}, B = \frac{-k_3}{2}$$

$$\sqrt{16-Ae^{Bt}} = P \ (P>0)$$

When $t=0, P=3$:

$$\sqrt{16-Ae^{B(0)}} = 3$$

$$16 - A = 3^2$$

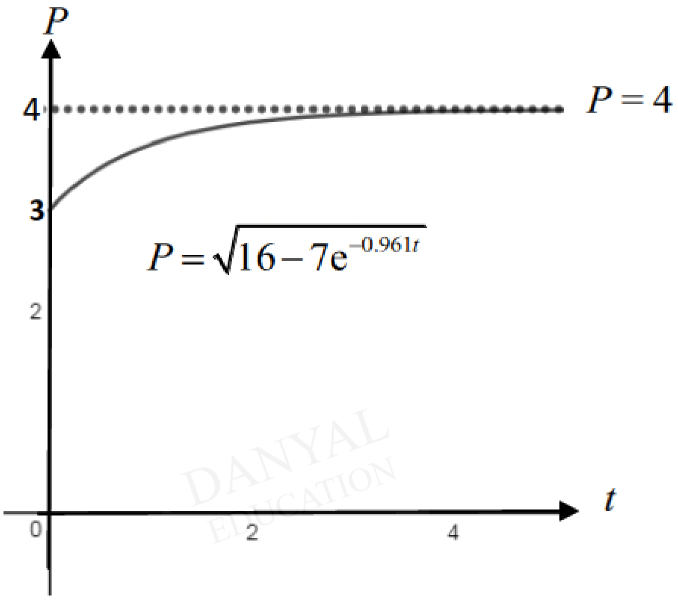
$$A = 7$$

When $t=1, P=3.65$:

$$\sqrt{16-7e^{B}} = 3.65$$

$$B = \ln\frac{16-3.65^2}{7} = -0.96102663 = -0.961$$

$$\therefore P = \sqrt{16-7e^{-0.961t}}$$



Rice production will only occur if the price is able to at least cover the initial cost of investment.

Q3

Q3

(i)

$\frac{\mathrm{d}M}{\mathrm{d}t} \propto I - kM$, where k is a positive constant.

$$\frac{\mathrm{d}M}{\mathrm{d}t} = b(I - kM)$$

If $I = 0$, $-\frac{1}{100}M = b(0 - kM)$

$$-\frac{M}{100} = -bkM$$

$$b = \frac{1}{100k}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{1}{100k}(I - kM) = \frac{I - kM}{100k}$$

$= \frac{I - aM}{100a}$, where $a = k$ (shown)

Assumption (any 1 below):

- The man does not exercise so that no food energy is used up through exercising.
- The man does not fall sick so that no food energy is used up to help him recover from his illness.
- The man does not consume weight enhancing/loss supplements that affect his food energy gain/loss other than maintaining the healthy functioning of his body and increasing body mass.

(ii) For $\frac{\mathrm{d}M}{\mathrm{d}t}$ to be zero, $I = aM$

(iii)

$$\int \frac{a}{I - aM}\,\mathrm{d}M = \int \frac{1}{100}\,\mathrm{d}t$$

$$-\ln\left|I - aM\right| = \frac{t}{100} + C$$

$$\ln\left|I - aM\right| = \frac{-t}{100} - C$$

$$I - aM = \pm \mathrm{e}^{\frac{-t}{100}} \mathrm{e}^{-C} = A\mathrm{e}^{\frac{-t}{100}}, \text{ where } A = \pm \mathrm{e}^{-C}$$

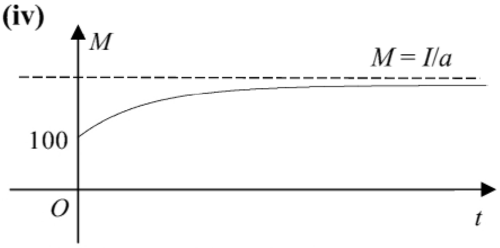
When $t = 0$, $M = 100 \Rightarrow A = I - 100a$

$$I - aM = (I - 100a)\mathrm{e}^{\frac{-t}{100}}$$

$$aM = I - (I - 100a)\mathrm{e}^{\frac{-t}{100}}$$

$$M = \frac{I}{a} - \left(\frac{I}{a} - 100\right)\mathrm{e}^{\frac{-t}{100}}$$

(iv)



Explanation (any 1 below):

- The man consumes more food than is necessary for maintaining a healthy functioning body. Therefore the graph shows that his body mass will increase.
- Since $I > 100a$, hence $\frac{I}{a} > 100$. The man's body mass is always less than $\frac{I}{a}$.
  In the long run, the man's body mass will approach $\frac{I}{a}$.

(v)

Given $I = 50a$,

$$90 = 50 - (50 - 100)\mathrm{e}^{\frac{-t}{100}}$$

Using equation found in **(iii)**

$$50\mathrm{e}^{\frac{-t}{100}} = 40$$

$$\mathrm{e}^{\frac{-t}{100}} = \frac{4}{5}$$

$$\frac{-t}{100} = \ln\frac{4}{5}$$

$$\therefore\ t = -100\ln\frac{4}{5} = 22.3 \text{ days} \quad (3 \text{ s.f.})$$