A Level H2 Math

Differential Equations Test 1

Q1

(i) Show that for any real constant k,

$$\int t^2 \mathrm{e}^{-kt}\,\mathrm{d}t = -\mathrm{e}^{-kt}\left(\frac{a}{k}t^2+\frac{b}{k^2}t+\frac{c}{k^3}\right)+D,$$

where D is an arbitrary constant, and a, b, and c are constants to be determined. [3]

On the day of the launch of a new mobile game, there were 100,000 players. After t months, the number of players on the game is x, in hundred thousands, where x and t are continuous quantities. It is known that, on average, one player recruits 0.75 players into the game per month, while the number of players who leave the game per month is proportional to t^2.

(ii) Write down a differential equation relating x and t. [1]

(iii) Using the substitution $x = u\mathrm{e}^{\frac{3}{4}t}$, show that the differential equation in **(ii)** can be reduced to

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -pt^2\mathrm{e}^{-\frac{3}{4}t},$$

where p is a positive constant.

Hence solve the differential equation in **(ii)**, leaving your answer in terms of p. [5]

(iv) For $p = \frac{1}{3}$, find the maximum number of players on the game, and determine if there will be a time when there are no players on the game. [2]

(v) Find the range of values of p such that the game will have no more players after some time. [2]

Q2

A drug is administered by an intravenous drip. The drug concentration, x, in the blood is measured as a fraction of its maximum level. The drug concentration after t hours is modelled by the differential equation

$$\frac{dx}{dt} = k\left(1+x-2x^2\right),$$

where $0 \le x < 1$, and k is a positive constant. Initially, $x = 0$.

(i) Find an expression for x in terms of k and t. [5]

After one hour, the drug concentration reaches 75% of its maximum level.

(ii) Show that the exact value of k is $\frac{1}{3}\ln 10$, and find the time taken for the drug concentration to reach 90% of its maximum level. [3]

A second model is proposed with the following differential equation

$$\frac{dx}{dt} = \sin^2\left(\frac{1}{2}t\right),$$

where x is the drug concentration, measured as a fraction of its maximum level, in the blood after t hours. Initially, $x = 0$.

(iii) Find an expression for x in terms of t. [3]

(iv) Explain, with the aid of a sketch, why this proposed second model is inappropriate. [2]

Q3

In a farm, the growth of the population of prawns is studied.

(a) The population of prawns of size n thousand at time t months satisfies the differential equation

$$\frac{d^2n}{dt^2} = e^{-\frac{t}{5}}.$$

(i) Find the general solution of this differential equation. [2]

(ii) It is given that initially, the size of the population of prawns is 50 000. Sketch on a single diagram, two distinct solution curves for the differential equation to illustrate the following two cases for large values of t :

I. the size of the population of prawns increases indefinitely,

II. the size of the population of prawns stabilizes at a certain positive number. [3]

(b) In order for the prawns to grow faster and be more resistance to diseases, a drug is administered to the prawns. The prawn's body metabolizes (breaks down) the drug at a rate proportional to the amount of drug, x mg, present in the body at time t hours.

(i) Given that the initial dosage is 0.1 mg, show that $x = \frac{1}{10}e^{-kt}$, where $k > 0$.

[4]

(ii) The half-life of a drug is defined as the time taken for half of it to be metabolized. Given that the half-life of this drug is 4 hours, find the exact value of k. [2]

(iii) If 0.1 mg of this drug is administered to the prawn every 8 hours, show that the total amount of drug present in the prawn's body at any time t is always less than 0.15 mg. [3]

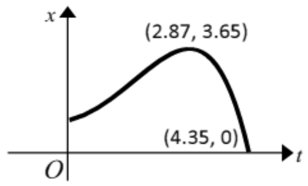
Answers

Differential Equations Test 1

Q1

1(i)	$\int t^2 e^{-kt}\, dt = -\frac{1}{k}e^{-kt}(t^2) - \int -\frac{1}{k}e^{-kt}(2t)\, dt$ $= -\frac{1}{k}t^2 e^{-kt} + \frac{2}{k}\left[-\frac{1}{k}e^{-kt}(t) - \int -\frac{1}{k}e^{-kt}(1)\, dt\right]$ $= -\frac{1}{k}t^2 e^{-kt} - \frac{2}{k^2}t e^{-kt} - \frac{2}{k^3}e^{-kt} + D$ $= -e^{-kt}\left(\frac{1}{k}t^2 + \frac{2}{k^2}t + \frac{2}{k^3}\right) + D$	Some students were careless in the first step and could only be awarded the subsequent method mark if they proceeded to integrate by parts a second time. Some students integrated the terms incorrectly or made wrong choices for the terms. Students should remember that the aim of integration by parts is to obtain a simpler integral which can then be integrated (unless it requires the "loop" technique which is not the case for this question) and realise that something is wrong if they ended up with one which looks even more complicated. Few students left this part blank or did not proceed to do integration by parts a second time. Quite a number of students did not put the final expression in the required form and lost marks. Students are reminded to take note of the requirements of the questions.
(ii)	$\frac{dx}{dt} = \frac{3}{4}x - pt^2$	Majority could not get this expression or even gave an expression for x in terms of t instead ($\frac{dx}{dt}$ was not even seen) which should not be the case since the question asked for a "differential equation". Some students also made mistakes in the unit for x (in hundred thousands) or missed out the "x" in the "$0.75x$" term (or incorrectly wrote it as $0.75t$) or missed out the constant of proportionality "p".

(iii)	$x=u\,e^{\frac{3}{4}t}\Rightarrow\frac{dx}{dt}=\frac{3}{4}u\,e^{\frac{3}{4}t}+e^{\frac{3}{4}t}\frac{du}{dt}$ $\frac{3}{4}u\,e^{\frac{3}{4}t}+e^{\frac{3}{4}t}\frac{du}{dt}=\frac{3}{4}u\,e^{\frac{3}{4}t}-pt^2\Rightarrow\frac{du}{dt}=-pt^2\,e^{-\frac{3}{4}t}$ $u=p\,e^{-\frac{3}{4}t}\left(\frac{1}{\frac{3}{4}}t^2+\frac{2}{\left(\frac{3}{4}\right)^2}t+\frac{2}{\left(\frac{3}{4}\right)^3}\right)+D$ $=p\,e^{-\frac{3}{4}t}\left(\frac{4}{3}t^2+\frac{32}{9}t+\frac{128}{27}\right)+D$ $\Rightarrow\frac{x}{e^{\frac{3}{4}t}}=p\,e^{-\frac{3}{4}t}\left(\frac{4}{3}t^2+\frac{32}{9}t+\frac{128}{27}\right)+D$ $\therefore x=p\left(\frac{4}{3}t^2+\frac{32}{9}t+\frac{128}{27}\right)+D\,e^{\frac{3}{4}t}$ When $t=0, x=1$, $1=p\left(\frac{128}{27}\right)+D\Rightarrow D=1-\frac{128}{27}p$ $x=p\left(\frac{4}{3}t^2+\frac{32}{9}t+\frac{128}{27}\right)+\left(1-\frac{128}{27}p\right)e^{\frac{3}{4}t}$	Students would not be able to show the given differential equation if the expression in (i) was incorrect. Some students were not able to correctly differentiate $u\,e^{\frac{3}{4}t}$. Students should read the question carefully and if they are not able to show the required DE, students should still proceed to solve the given DE, and not solve their own incorrect DE, which was what many students did. Many students incorrectly used $k=-\frac{3}{4}$ and were penalised. A few students failed to see the link to part (i) and redid the integration without using the results obtained in (i). Many students failed to substitute "x" back into the solution and of those who did, majority forgot the arbitrary constant D or forgot to multiply $e^{\frac{3}{4}t}$ to D – some even labelled $D\,e^{\frac{3}{4}t}$ as another constant $E=D\,e^{\frac{3}{4}t}$ which is incorrect since it now contains the variable *t* and is not just a product of constants. Many also failed to sub in the initial conditions, which was required to obtain the arbitrary constant in terms of *p*. Some did so in the next part but no credit was awarded since it was the requirement in (iii). Some students used the wrong units or failed to show the link from *x* to *u* when using the initial conditions.
(iv)	When $p=\frac{1}{3}$,	Parts (iv) and (v) were badly

	$x=\frac{1}{3}\left(\frac{4}{3}t^2+\frac{32}{9}t+\frac{128}{27}\right)+\left(-\frac{47}{81}\right)e^{\frac{3}{4}t}$  Maximum number of players on the game = 365 000. Yes, $x = 0$ when $t = 4.35$ months.	done as students were not likely to obtain the answers to these parts if their expression for x was incorrectly in (iii) – only a handful of students could obtain the correct final expression for x in (iii). Students were expected to use the GC (graph) for this part and not expected to differentiate, solve the equation, etc to find the maximum value or the t-value which gave 0 players - only 2 marks are awarded for the two required answers and students can get the hint from the marks allocation that they were not expected to manually find these answers on their own.
(v)	For $x=0$ after some time, $1-\frac{128}{27}p<0 \Rightarrow p>\frac{27}{128}=0.211$	This required students to see that the coefficient of the exponential term, $\left(1-\frac{128}{27}p\right)e^{\frac{3}{4}t}$, in the expression for x found in (iii) had to be negative in order for there to be no players after some time. However, as mentioned in (iii), only a handful of students had the exponential term in their solution for x, thus this part was not well done.

Q2

(i) **Method 1: Using Partial Fractions**

$$\frac{1}{1+x-2x^2}\frac{dx}{dt}=k$$

$$\int \frac{1}{1+x-2x^2}\ dx=\int k\ dt$$

$$\frac{2}{3}\int \frac{1}{2x+1}\ dx-\frac{1}{3}\int \frac{1}{x-1}\ dx=\int k\ dt$$

$$\frac{1}{1+x-2x^2}=\frac{1}{(1-x)(1+2x)}=\frac{2/3}{2x+1}-\frac{1/3}{x-1}$$

$$\frac{1}{3}\ln|2x+1|-\frac{1}{3}\ln|x-1|=kt+C$$

$$\frac{1}{3}\ln\left|\frac{2x+1}{x-1}\right|=kt+C$$

$$\frac{2x+1}{x-1}=Ae^{3kt},\ A=\pm e^{3C}$$

$$x=\frac{Ae^{3kt}+1}{Ae^{3kt}-2}$$

When $t=0,\ x=0$: $\quad 0=\dfrac{A+1}{A-2}\Rightarrow A=-1$

$$\therefore x=\frac{e^{3kt}-1}{e^{3kt}+2}$$

Method 2: Completing the square

$$\frac{1}{1+x-2x^2}\frac{dx}{dt}=k$$

$$\int \frac{1}{1+x-2x^2}\ dx=\int k\ dt$$

$$\int \frac{1}{-2\left(x-\frac{1}{4}\right)^2+\frac{9}{8}}\ dx=\int k\ dt$$

$$\frac{1}{2}\int \frac{1}{\left(\frac{3}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}\ dx=\int k\ dt$$

$$\frac{1}{2}\left(\frac{1}{2\left(\frac{3}{4}\right)}\right)\ln\left|\frac{\frac{3}{4}+x-\frac{1}{4}}{\frac{3}{4}-\left(x-\frac{1}{4}\right)}\right|=kt+C$$

$$\frac{1}{3}\ln\left|\frac{\frac{1}{2}+x}{1-x}\right|=kt+C$$

$$\frac{1}{3}\ln\left|\frac{2x+1}{2(1-x)}\right|=kt+C$$

$$\frac{2x+1}{2(1-x)}=Ae^{3kt},\ A=\pm e^{3C}$$

$$x=\frac{2Ae^{3kt}-1}{2\left(Ae^{3kt}+1\right)}$$

When $t=0,\ x=0:\quad 0=\dfrac{2A-1}{2(A+1)} \Rightarrow A=\dfrac{1}{2}$

$\therefore x=\dfrac{e^{3kt}-1}{e^{3kt}+2}$

(ii)

When $t=1,\ x=\dfrac{3}{4}:\quad \therefore \dfrac{3}{4}=\dfrac{e^{3k}-1}{e^{3k}+2} \Rightarrow e^{3k}=10$

$\Rightarrow k=\dfrac{1}{3}\ln 10 \ \text{(shown)}$

$\therefore x=\dfrac{10^t-1}{10^t+2}$

When $x=\dfrac{9}{10}:\quad \therefore \dfrac{9}{10}=\dfrac{10^t-1}{10^t+2} \Rightarrow 10^t=28$

$\Rightarrow t=\dfrac{\ln 28}{\ln 10}$

$= 1.45$ hours (3 s.f.)

Also Accept: 86.8 mins (3 s.f.)

(iii)

$\dfrac{dx}{dt}=\sin^2\left(\dfrac{1}{2}t\right)$

$=\dfrac{1}{2}-\dfrac{1}{2}\cos t$

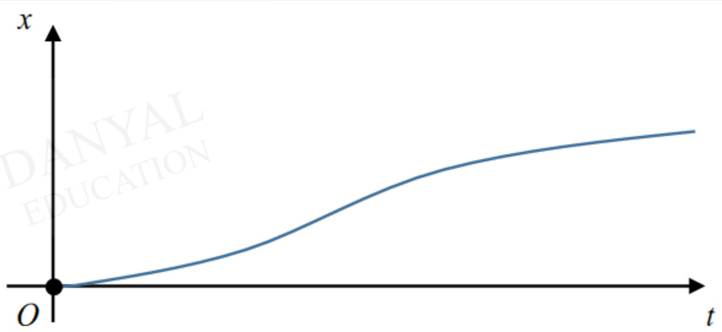
$x=\displaystyle\int \dfrac{1}{2}-\dfrac{1}{2}\cos t \, dt$

$=\dfrac{1}{2}t-\dfrac{1}{2}\sin t+C$

When $t=0,\ x=0:\quad C=0$

$\therefore x=\dfrac{1}{2}t-\dfrac{1}{2}\sin t$

(iv)



The graph shows that as time increases, the drug concentration still continue to increase / the curve shows a strictly increasing function beyond the maximum level of drug concentration.

Q3

(ai)

Let n denote the population of prawns in thousands at time t

$$\frac{d^2 n}{dt^2} = e^{-\frac{t}{5}}$$

$$\frac{dn}{dt} = -5e^{-\frac{t}{5}} + C$$

$$n = 25e^{-\frac{t}{5}} + Ct + D$$

(aii)

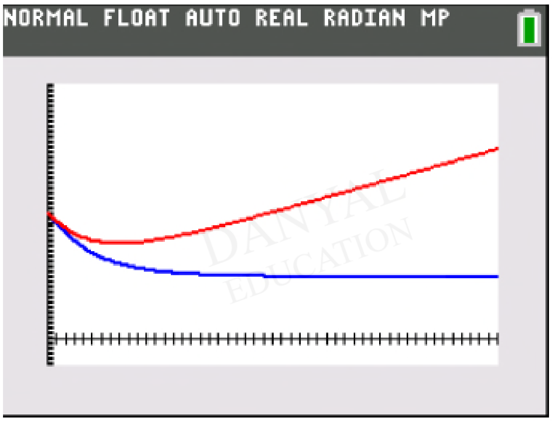
Given $n = 50,\ t = 0,$

$50 = 25 + D \Rightarrow D = 25$

$n = 25e^{-\frac{t}{5}} + Ct + 25$

I Requires $C > 0$ so that $n = 25e^{-\frac{t}{5}} + Ct + 25 \to \infty$ as $t \to \infty$

II Requires $C = 0$ so that $n = 25e^{-\frac{t}{5}} + 25$
Then as $t \to \infty$, $n \to 25$



(bi)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -kx$$

$$\int \frac{1}{x}\ \mathrm{d}x = -k\int 1\ \mathrm{d}t$$

$$\ln|x| = -kt + C$$

$x = A\mathrm{e}^{-kt}$ where $A = \pm\mathrm{e}^{C}$

At $t = 0$, $x = 0.1$,

$\therefore A = 0.1$

$x = \frac{1}{10}\mathrm{e}^{-kt}$ (shown)

(bii)

At $t = 4$, $x = 0.05$,

$\therefore 0.05 = 0.1\mathrm{e}^{-4k}$

$\Rightarrow \mathrm{e}^{-4k} = \frac{1}{2}$

$\Rightarrow -k = \frac{\ln\frac{1}{2}}{4}$

$\Rightarrow k = -\frac{1}{4}\ln\frac{1}{2} = \frac{\ln 2}{4}$

(biii)

Total amount of drug present in the prawn's body at any time t

$< 0.1 + 0.1\mathrm{e}^{-\left(\frac{\ln 2}{4}\right)8} + 0.1\mathrm{e}^{-2\left(\frac{\ln 2}{4}\right)8} + 0.1\mathrm{e}^{-3\left(\frac{\ln 2}{4}\right)8} + \ldots$

$= \frac{0.1}{1 - \mathrm{e}^{-\left(\frac{\ln 2}{4}\right)8}}$

$= \frac{2}{15} < 0.15$

$\therefore$ The total amount of drug present in the prawn's body at any time t is always less than 0.15 mg.