A Level H2 Math

Complex Numbers Test 9

Q1

(a) In an Argand diagram, points P and Q represent the complex numbers $z_1 = 2+3\text{i}$ and $z_2 = \text{i}z_1$.

(i) Find the area of the triangle OPQ, where O is the origin. [2]

(ii) z_1 and z_2 are roots of the equation $(z^2+az+b)(z^2+cz+d)=0$, where $a, b, c, d \in \mathbb{R}$. Find a, b, c and d. [4]

(b) Without using the graphing calculator, find in exact form, the modulus and argument of $v^* = \left(\frac{\sqrt{3}+\text{i}}{-1+\text{i}}\right)^{14}$. Hence express v in exponential form. [5]

Q2

It is given that $z=\sqrt{3}+\text{i}$ and $w=-1+\text{i}$.

(i) Without using a calculator, find an exact expression for $\frac{z^2}{w^*}$. Give your answer in the form $r\text{e}^{\text{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [4]

(ii) Find the exact value of the real number q such that $\arg\left(1-\frac{q}{z}\right)=\frac{\pi}{12}$. [3]

Q3

Do not use a calculator in answering this question.

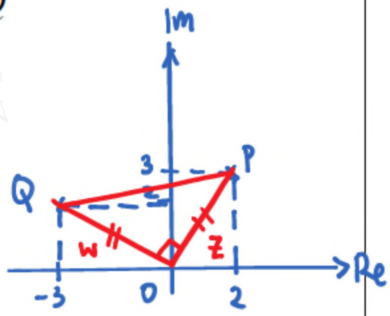
Given that $z=1+\text{i}$ is a root of the equation $2z^4+az^3+7z^2+bz+2=0$, find the values of the real numbers a and b and the other roots. [5]

Deduce the roots of the equation $2z^4+bz^3+7z^2+az+2=0$. [2]

Answers

Complex Numbers Test 9

Q1

(a)(i) Since $w = \mathrm{i}z$, then $OP \perp OQ$ i.e. $\angle POQ = 90^o$. Area of triangle OPQ $= \frac{1}{2}\lvert z \rvert \lvert w \rvert$ $= \frac{1}{2}\lvert 2+3\mathrm{i} \rvert^2$ $= \frac{13}{2}$ units^2 	
(a)(ii) Since $(z^2+az+b)(z^2+cz+d)=0$ is a polynomial with constant coefficients, complex roots occur in conjugate pairs. Therefore, the four roots are $2+3\mathrm{i}$, $2-3\mathrm{i}$, $-3+2\mathrm{i}$ and $-3-2\mathrm{i}$. $[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})][z-(-3+2\mathrm{i})][z-(-3-2\mathrm{i})]$ $=(z^2-4z+13)(z^2+6z+13)$ Hence, $a=-4,\ b=13,\ c=6,\ d=13$.	Students should write out clearly the roots of the equation. There are 2 ways to expand $[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})]$ Method 1 $[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})]$ $=z^2-(2+3\mathrm{i}+2-3\mathrm{i})z+(2+3\mathrm{i})(2-3\mathrm{i})$ Method 2 $[z-(2+3\mathrm{i})][z-(2-3\mathrm{i})]$ $=[(z-2)-(3\mathrm{i})][(z-2)+(3\mathrm{i})]$ $=(z-2)^2-(3\mathrm{i})^2$

(b) $\lvert v^*\rvert = \dfrac{\lvert\sqrt{3}+\mathrm{i}\rvert^{14}}{\lvert -1+\mathrm{i}\rvert^{14}}$ $= \dfrac{2^{14}}{\left(\sqrt{2}\right)^{14}} = 2^7$ $\arg(v^*)$ $= \arg\left(\left(\dfrac{\sqrt{3}+\mathrm{i}}{-1+\mathrm{i}}\right)^{14}\right)$ $= 14\left[\arg\left(\sqrt{3}+\mathrm{i}\right) - \arg(-1+\mathrm{i})\right]$ $= 14\left[\dfrac{\pi}{6} - \dfrac{3\pi}{4}\right]$ $= -\dfrac{49\pi}{6} \notin (-\pi, \pi]$ $\therefore\ \arg(v^*) = -\dfrac{\pi}{6}$ $\Rightarrow \arg(v) = \dfrac{\pi}{6}$ Since $\lvert v\rvert = \lvert v^*\rvert = 2^7$, then $v = 2^7 \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$.	Need to take note of how to present $\arg(v^*)$.

Marker's comments

(a)(i) This part was not well answered. Many students who were unclear/not aware that *OP* is perpendicular to *OQ* had problem arriving at the correct answer for the area of triangle *OPQ*. Many students used a variety of method (using vectors and cross product, shoelace method, area of trapezium/area of triangles) to find the area of triangle, some with more success than others.

(ii) Although many students were able to recognize that the complex roots occur in conjugate pairs since the coefficients of the equation are real, many students were unable to pair the factors $\left(z-(2+3\mathrm{i})\right)$ and $\left(z-(2-3\mathrm{i})\right)$, $\left(z-(-3+2\mathrm{i})\right)$ and $\left(z-(-3-2\mathrm{i})\right)$. Errors also occurred during the expansion of $\left(z-(2+3\mathrm{i})\right)\left(z-(2+3\mathrm{i})\right)$ and $\left(z-(-3+2\mathrm{i})\right)\left(z-(-3-2\mathrm{i})\right)$. A handful of students substituted $(2+3\mathrm{i})$ into the equation and managed to find a,b,c,d by comparing real and imaginary parts. Those who were unsuccessful in this method will not gain marks.

(b) Despite the statement at the start of the question, a large number of students used their GC to obtain modulus and argument to parts of the question. Many students (about 80%) of students rationalize the expression $\left(\frac{\sqrt{3}+\mathrm{i}}{-1+\mathrm{i}}\right)$ and soon realized that they did not have much success to solve the question except to use GC to obtain the modulus and argument. A number of students wrote down what they believed to be the argument of $(-1+\mathrm{i})$ without considering where the complex number was on an Argand diagram.

This part clearly indicated that many students were weak in their understanding of the fundamental concepts of Complex Numbers.

Q2

(i)

$|z| = |\sqrt{3} + i| = \sqrt{3+1} = 2$,

$|w| = |-1 + i| = \sqrt{1+1} = \sqrt{2}$

$\arg(z) = \arg(\sqrt{3} + i)$

$= \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\arg(w) = \arg(-1 + i)$

$= \pi - \tan^{-1} 1 = \frac{3\pi}{4}$

$$\frac{z^2}{w^*} = \frac{\left(2e^{i\frac{\pi}{6}}\right)^2}{\sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)}}$$

$$= 2^{\frac{3}{2}} e^{i\frac{13\pi}{12}}$$

$$= 2^{\frac{3}{2}} e^{-i\frac{11\pi}{12}}$$

ii)

$$\arg\left(1 - \frac{q}{z}\right) = \arg\left(\frac{z-q}{z}\right)$$

$$= \arg(z-q) - \arg(z) = \frac{\pi}{12}$$

$$\arg(z-q) = \frac{\pi}{12} + \frac{\pi}{6} = \frac{\pi}{4}$$

$$\arg\left(\left(\sqrt{3} - q\right) + i\right) = \frac{\pi}{4}$$

$$\sqrt{3} - q = 1 \Rightarrow q = \sqrt{3} - 1$$

Q3

By Conjugate Root Theorem, $z=1-\mathrm{i}$ is also a root.

$$\left[z-(1+\mathrm{i})\right]\left[z-(1-\mathrm{i})\right]=\left[(z-1)-\mathrm{i}\right]\left[(z-1)+\mathrm{i}\right]$$
$$=(z-1)^2-\mathrm{i}^2$$
$$=z^2-2z+2$$

$$\left(z^2-2z+2\right)\left(Az^2+Bz+C\right)=2z^4+az^3+7z^2+bz+2$$

By observation, $A=2$, $C=1$.

i.e. $\left(z^2-2z+2\right)\left(2z^2+Bz+1\right)=2z^4+az^3+7z^2+bz+2$

Coeff. of z^2 : $1-2B+4=7 \Rightarrow B=-1$

Coeff. of z^3 : $B-4=a \Rightarrow a=-5$

Coeff. of z : $-2+2B=b \Rightarrow b=-4$

$$2z^2-z+1=0$$
$$z=\frac{1\pm\sqrt{1-4(2)}}{2(2)}$$
$$z=\frac{1\pm\sqrt{7}\mathrm{i}}{4}$$

Hence other roots are $1-\mathrm{i}$, $\dfrac{1\pm\sqrt{7}\mathrm{i}}{4}$.

$$2z^4+bz^3+7z^2+az+2=0$$
$$2+b\frac{1}{z}+7\frac{1}{z^2}+a\frac{1}{z^3}+2\frac{1}{z^4}=0$$

Hence

$$z=\frac{1}{1+\mathrm{i}},\quad \frac{1}{1-\mathrm{i}},\quad \frac{4}{1+\sqrt{7}\mathrm{i}},\quad \frac{4}{1-\sqrt{7}\mathrm{i}}$$
$$z=\frac{1-\mathrm{i}}{2},\quad \frac{1+\mathrm{i}}{2},\quad \frac{1-\sqrt{7}\mathrm{i}}{2},\quad \frac{1+\sqrt{7}\mathrm{i}}{2}$$