A Level H2 Math

Complex Numbers Test 8

Q1

Do not use a calculator in answering this question.

(i) Explain why the equation $z^3 + az^2 + az + 7 = 0$ cannot have more than two non-real roots, where a is a real constant. [1]

(ii) Given that $z = -7$ is a root of the equation in **(i)**, find the other roots, leaving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

(iii) Hence, solve the equation $iz^3 + 8z^2 - 8iz - 7 = 0$, leaving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]

Q2

Do not use a calculator in answering this question.

Showing your working, find the complex numbers z and w which satisfy the simultaneous equations

$$4iz - 3w = 1 + 5i \quad \text{and}$$

$$2z + (1+i)w = 2 + 6i. \qquad [5]$$

Q3

The complex number z has modulus 3 and argument $\frac{2\pi}{3}$.

(i) Find the modulus and argument of $\frac{-2\text{i}}{z^*}$, where z^* is the complex conjugate of z, leaving your answers in the exact form. [3]

(ii) Hence express $\frac{-2\text{i}}{z^*}$ in the form of $x+\text{i}y$, where x and y are real constants, giving the exact values of x and y in non-trigonometrical form. [2]

(iii) The complex number w is defined such that $w=1+\text{i}k$, where k is a non-zero real constant. Given that $\frac{-2\text{i}w}{z^*}$ is purely imaginary, find the exact value of k. [2]

Answers

Complex Numbers Test 8

Q1

(i) Since a is real, the polynomial equation has real coefficients, and thus all non-real roots must be in conjugate pairs. Since the degree of the polynomial is three, there will be 3 roots. The highest even number below 3 is 2.

(ii)

$$z^3 + az^2 + az + 7 = 0$$

$$(-7)^3 + a(-7)^2 + a(-7) + 7 = 0$$

$$a = 8$$

$$z^3 + 8z^2 + 8z + 7 = 0$$

$$(z+7)(z^2 + z + 1) = 0$$

$$z = -7 \text{ or } z = \frac{-1 \pm i\sqrt{3}}{2}$$

$$z = 7e^{i\pi},\ e^{\frac{i2\pi}{3}}, e^{-\frac{i2\pi}{3}}$$

(iii)

$$-iz^3 - 8z^2 + 8iz + 7 = 0$$

$$(iz)^3 + 8(iz)^2 + 8(iz) + 7 = 0$$

From (ii), replace z with iz

$$iz = 7e^{i\pi},\ e^{\frac{i2\pi}{3}}, e^{-\frac{i2\pi}{3}}$$

$$\Rightarrow z = -i7e^{i\pi},\ -ie^{\frac{i2\pi}{3}}, -ie^{-\frac{i2\pi}{3}}$$

$$\Rightarrow z = e^{-\frac{i\pi}{2}}7e^{i\pi},\ e^{-\frac{i\pi}{2}}e^{\frac{i2\pi}{3}}, e^{-\frac{i\pi}{2}}e^{-\frac{i2\pi}{3}}$$

$$\Rightarrow z = 7e^{\frac{i\pi}{2}},\ e^{\frac{i\pi}{6}}, e^{\frac{i5\pi}{6}}$$

Q2

$$4iz-3w=1+5i \text{ --------(1)}$$

$$2z+(1+i)w=2+6i \text{ --------(2)}$$

$$(2)\times 2i$$

$$4iz+2i(1+i)w=2i(2+6i)$$

$$4iz+2iw-2w=4i-12 \text{ --------(3)}$$

$$(3)-(1):$$

$$4iz+2iw-2w-(4iz-3w)=(4i-12)-(1+5i)$$

$$w+2iw=-13-i$$

$$(1+2i)w=-13-i$$

$$w=\left(\frac{-13-i}{1+2i}\right)\left(\frac{1-2i}{1-2i}\right)$$

$$w=\frac{-13+26i-i-2}{(1)^2-(2i)^2}$$

$$w=\frac{-15+25i}{5}$$

$$w=-3+5i$$

Substitute $w=-3+5i$ into (2)

$$2z=2+6i-(1+i)(-3+5i)$$

$$2z=2+6i-(-3+5i-3i-5)$$

$$2z=2+6i-(-8+2i)$$

$$2z=10+4i$$

$$z=5+2i$$

$\therefore w=-3+5i$ and $z=5+2i$.

Q3

(i) Given $|z| = 3,\ \arg(z) = \frac{2\pi}{3}$,

$$\left|\frac{-2\mathrm{i}}{z^*}\right| = \frac{|-2i|}{|z^*|} = \frac{2}{3} \quad (\because |z| = |z^*|)$$

$$\arg\left(\frac{-2\mathrm{i}}{z^*}\right) = \arg(-2\mathrm{i}) - \arg(z^*)$$

$$= -\frac{\pi}{2} - \left(-\frac{2\pi}{3}\right) \quad (\because \arg(z^*) = -\arg(z))$$

$$= \frac{\pi}{6}$$

(ii)

$$\frac{-2\mathrm{i}}{z^*} = \frac{2}{3}\left[\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{2}{3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$$

$$= \frac{\sqrt{3}}{3} + \frac{1}{3}\mathrm{i}$$

(iii)

$$\frac{-2\mathrm{i}w}{z^*} = \left(\frac{\sqrt{3}}{3} + \frac{1}{3}\mathrm{i}\right)(1 + \mathrm{i}k)$$

$$= \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}k\mathrm{i} + \frac{1}{3}\mathrm{i} - \frac{1}{3}k$$

Since $\frac{-2\mathrm{i}w}{z^*}$ is purely imaginary,

$$\frac{\sqrt{3}}{3} - \frac{1}{3}k = 0$$

$$k = \sqrt{3}$$