A Level H2 Math

Complex Numbers Test 7

Q1

(i) Show that for any complex number $z = re^{i\theta}$, where $r > 0$, and $-\pi < \theta \leq \pi$,

$$\frac{z}{z-r} = \frac{1}{2} - \frac{1}{2}\left(\cot\frac{\theta}{2}\right)i.$$

[3]

(ii) Given that $z = 2e^{i\left(\frac{\pi}{3}\right)}$ is a root of the equation $z^2 - 2z + 4 = 0$. State, in similar form, the other root of the equation. **[1]**

(iii) Using parts **(i)** and **(ii)**, solve the equation $\frac{4w^2}{(w-1)^2} - \frac{4w}{w-1} + 4 = 0$. **[4]**

Q2

Without the use of a calculator, find the complex numbers z and w which satisfy the simultaneous equations

$$z - wi = 3$$
$$z^2 - w + 6 + 3i = 0$$

[6]

Q3

It is given that z_1, z_2 and z_3 are the roots of the equation

$$2z^3 + pz^2 + qz - 4 = 0$$

such that $\arg z_1 < \arg z_2 < \arg z_3$ and $z_1 = 1 - i\sqrt{3}$. Find the values of the real numbers p and q. [3]

(i) Without using the calculator, find z_2 and z_3. [3]

In an Argand diagram, points P, Q and R represent the complex numbers z_1, $w = \sqrt{2} + i\sqrt{2}$ and $z_1 + w$ respectively and O is the origin.

(ii) Express each of z_1 and w in the form $r e^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give r and θ in exact form. [2]

(iii) Indicate P, Q and R on the Argand diagram and identify the type of the quadrilateral $OPRQ$. [3]

(iv) Find the exact value of $\arg\left(z_1^{\,4} w^*\right)$. [3]

Answers

Complex Numbers Test 7

Q1

(i)

$$\frac{re^{i\theta}}{re^{i\theta}-r}$$

$$=\frac{e^{i\theta}}{e^{i\left(\frac{\theta}{2}\right)}\left(e^{i\left(\frac{\theta}{2}\right)}-e^{-i\left(\frac{\theta}{2}\right)}\right)}$$

$$=\frac{e^{i\left(\frac{\theta}{2}\right)}}{2i\sin\left(\frac{\theta}{2}\right)}$$

$$=\frac{\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)}{2i\sin\left(\frac{\theta}{2}\right)}$$

$$=\frac{1}{2}+\frac{1}{2i}\cot\left(\frac{\theta}{2}\right)$$

$$=\frac{1}{2}-\frac{1}{2}\left(\cot\left(\frac{\theta}{2}\right)\right)i$$

(ii)

$$z=2e^{i\left(\frac{\pi}{3}\right)}$$

(iii)

$$\frac{4w^2}{(w-1)^2}-\frac{4w}{w-1}+4=0$$

$$\left(\frac{2w}{w-1}\right)^2-2\left(\frac{2w}{w-1}\right)+4=0$$

Let $z=\frac{2w}{w-1}$, then

$$z^2-2z+4=0$$

From (ii) the solutions are $z=2e^{i\left(\frac{\pi}{3}\right)}$ or $z=2e^{-i\left(\frac{\pi}{3}\right)}$

Since

$$z=\frac{2w}{w-1}$$

$$zw-z=2w$$

$$w(z-2)=z$$

$$w=\frac{z}{z-2}$$

Part (i) result can be used as $z=2e^{i\left(\frac{\pi}{3}\right)}$, where $r=2$ with $\theta=\frac{\pi}{3},\ \theta=-\frac{\pi}{3}$.

$$w=\frac{1}{2}-\frac{1}{2}\left(\cot\frac{\pi}{6}\right)i \quad \text{or} \quad w=\frac{1}{2}-\frac{1}{2}i\cot\left(-\frac{\pi}{6}\right)$$

$$w=\frac{1}{2}-\frac{\sqrt{3}}{2}i \text{ or } w=\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

Q2

Method 1

$z - w\text{i} = 3$

$\Rightarrow \quad w = \dfrac{z-3}{\text{i}} = 3\text{i} - z\text{i}$ --- (1)

Substitute (1) into $z^2 - w + 6 + 3\text{i} = 0$

$z^2 - (3\text{i} - z\text{i}) + 6 + 3\text{i} = 0$

$\Rightarrow \; z^2 + z\text{i} + 6 = 0$

$\Rightarrow \; z = \dfrac{-\text{i} \pm \sqrt{(\text{i})^2 - 4(1)(6)}}{2} = \dfrac{-\text{i} \pm \sqrt{-1-24}}{2}$

$= \dfrac{-\text{i} \pm 5\text{i}}{2}$

$\therefore \; z = 2\text{i}$ or $z = -3\text{i}$

$\Rightarrow w = 3\text{i} - (2\text{i})\text{i}$ $\qquad$ $w = 3\text{i} - (-3\text{i})\text{i}$

$= 2 + 3i$ $\qquad$ $= -3 + 3\text{i}$

Method 2

$z - w\text{i} = 3$

$\Rightarrow \quad z = 3 + w\text{i}$ …. (1)

Substitute (1) into $z^2 - w + 6 + 3\text{i} = 0$

$(3 + w\text{i})^2 - w + 6 + 3\text{i} = 0$

$\Rightarrow \quad 9 + 6w\text{i} - w^2 - w + 6 + 3\text{i} = 0$

$\Rightarrow \quad -w^2 - (1 - 6\text{i})w + 15 + 3\text{i} = 0$

$\Rightarrow \quad -w^2 - (1 - 6\text{i})w + 15 + 3\text{i} = 0$

$\Rightarrow \quad w^2 + (1 - 6\text{i})w - 15 - 3\text{i} = 0$

$\Rightarrow \quad w = \dfrac{-(1-6\text{i}) \pm \sqrt{(1-6\text{i})^2 - 4(1)(-15-3\text{i})}}{2}$

$= \dfrac{-1 + 6\text{i} \pm \sqrt{1 - 12\text{i} - 36 + 60 + 12\text{i}}}{2}$

$= \dfrac{-1 + 6\text{i} \pm \sqrt{25}}{2}$

$\therefore \; w = 2 + 3\text{i}$ or $w = -3 + 3\text{i}$

$\Rightarrow \; z = 3 + (2 + 3\text{i})\text{i} = 2\text{i}$ $\qquad$ $z = 3 + (-3 + 3\text{i})\text{i} = -3\text{i}$

Method 3

$w\text{i} = z - 3$

$\Rightarrow w = -\text{i}z + 3\text{i}$

$\therefore \; z^2 - (-\text{i}z + 3\text{i}) + 6 + 3\text{i} = 0$

$\Rightarrow \; z^2 + \text{i}z + 6 = 0$

Let $z = a + b\text{i}$ where $a, b \in \mathbb{R}$

$(a + b\text{i})^2 + \text{i}(a + b\text{i}) + 6 = 0$

$\Rightarrow \; a^2 - b^2 + 2ab\text{i} + a\text{i} - b + 6 = 0$

$\Rightarrow \; a^2 - b^2 - b + 6 + (2ab + a)\text{i} = 0$

By comparing the real and imaginary parts,

$a^2 - b^2 - b + 6 = 0 \quad \ldots (1)$

$2ab + a = 0 \qquad\qquad \ldots (2)$

From (2), $a = 0$ or $b = -\frac{1}{2}$

When $a = 0$, $b^2 + b - 6 = 0$

$(b-2)(b+3) = 0$

$b = 2$ or $b = -3$

Hence $z = 2\text{i}, w = -\text{i}(2\text{i}) + 3\text{i} = 2 + 3\text{i}$

or $z = -3\text{i}, w = -\text{i}(-3\text{i}) + 3\text{i} = -3 + 3\text{i}$

When $b = -\frac{1}{2}$, $a^2 = \frac{1}{4} - \frac{1}{2} - 6 = -\frac{25}{4}$

There is no real solution for a.

Q3

)

Since $1-\sqrt{3}\text{i}$ is a root,

$$2\left(1-\text{i}\sqrt{3}\right)^3+p\left(1-\text{i}\sqrt{3}\right)^2+q\left(1-\text{i}\sqrt{3}\right)-4=0$$

$$2(-8)+p\left(-2-2\sqrt{3}\text{i}\right)+q\left(1-\text{i}\sqrt{3}\right)-4=0$$

$$(-20-2p+q)+\left(-2\sqrt{3}p-\sqrt{3}q\right)\text{i}=0$$

Compare real and imaginary parts:

$$-2p+q=20 \quad ---(1)$$

$$-2\sqrt{3}p-\sqrt{3}q=0 \quad ---(2)$$

$$\therefore p=-5, \quad q=10$$

i

Since $1-\sqrt{3}\text{i}$ is a root, and all coefficients are real

$\Rightarrow 1+\sqrt{3}\text{i}$ is also a root.

$$2z^3-5z^2+10z-4=\left(z-\left(1-\sqrt{3}\text{i}\right)\right)\left(z-\left(1+\sqrt{3}\text{i}\right)\right)(2z+a)$$

$$=\left(z^2-2z+4\right)(2z+a)$$

By observation: $a=-1$

$$\therefore z_2=\frac{1}{2}, \quad z_3=1+\sqrt{3}\text{i}$$

ii

$|z_1|=\sqrt{1+3}=2$

$$\arg z_1=\arg\left(1-\sqrt{3}\text{i}\right)$$

$$=-\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$=-\frac{\pi}{3}$$

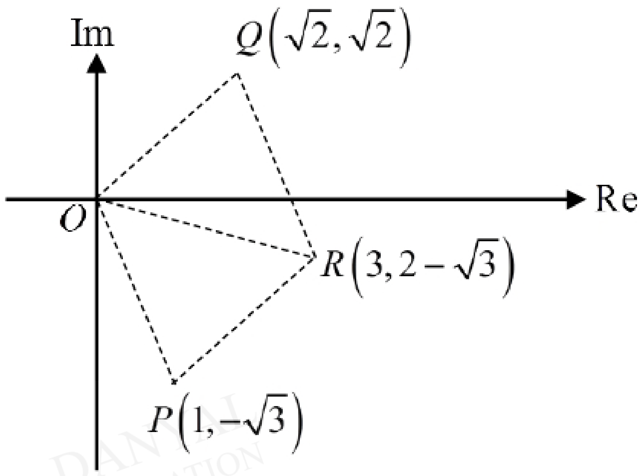
$|w|=\sqrt{2+2}=2$

$$\arg w=\arg\left(\sqrt{2}+\text{i}\sqrt{2}\right)$$

$$=\frac{\pi}{4}$$

$$\therefore z_1=2e^{-\frac{\pi}{3}\text{i}}, \quad w=2e^{\frac{\pi}{4}\text{i}}$$

iii



Quadrilateral $OPRQ$ is a rhombus

iv

$$4\arg\left(z_1\right)+\arg\left(w^*\right)=4\arg\left(z_1\right)-\arg\left(w\right)$$
$$=-\frac{4\pi}{3}-\frac{\pi}{4}$$
$$=-\frac{19\pi}{12}$$

$$\arg\left(z_1{}^4w^*\right)=-\frac{19\pi}{12}+2\pi$$
$$=\frac{5\pi}{12}$$