A Level H2 Math

Complex Numbers Test 6

Q1

A graphic calculator is **not** to be used in answering this question.

(a) The equation $w^3 + pw^2 + qw + 30 = 0$, where p and q are real constants, has a root $w = 2 - \mathrm{i}$. Find the values of p and q, showing your working. [3]

(b) The equation $z^2 + (-5 + 2\mathrm{i})z + (21 - \mathrm{i}) = 0$ has a root $z = 3 + u\mathrm{i}$, where u is real constant. Find the value of u and hence find the second root of the equation in cartesian form, $a + b\mathrm{i}$, showing your working. [5]

Q2

The complex number z is such that $|z| = 1$ and $\arg z = \theta$, where $0 < \theta < \frac{\pi}{4}$.

(i) Mark a possible point A representing z on an Argand diagram. Hence, mark the points B and C representing z^2 and $z + z^2$ respectively on the same Argand diagram corresponding to point A. [2]

(ii) State the geometrical shape of $OACB$. [1]

(iii) Express $z + z^2$ in polar form, $p\cos(q\theta)\left[\cos(k\theta) + \mathrm{i}\sin(k\theta)\right]$, where p, q and k are constants to be determined. [2]

Q3

Do not use a calculator in answering this question.

(a) One root of the equation $z^4+2z^3+az^2+bz+50=0$, where a and b are real, is $z=1+$

(i) Show that $a=7$ and $b=30$ and find the other roots of the equation. [5]

(ii) Deduce the roots of the equation $w^4-2\mathrm{i}w^3-7w^2+30\mathrm{i}w+50=0$. [2]

(b) Given that $p^*=\dfrac{\left(-\frac{1}{\sqrt{3}}+\mathrm{i}\right)^5}{(1-\mathrm{i})^4}$, by considering the modulus and argument of p^*, find the exact expression for p, in cartesian form $x+\mathrm{i}y$. [4]

* Please note: Part (a), the root given is z = 1 + 3i

Answers

Complex Numbers Test 6

Q1

(a)

Method 1

Since the coefficients are real, $w=2+\mathrm{i}$ is another root of the equation.

$$(w-2+\mathrm{i})(w-2-\mathrm{i})=(w-2)^2-(\mathrm{i})^2$$
$$=w^2-4w+4+1$$
$$=w^2-4w+5$$

$w^3+pw^2+qw+30=0$

$\left(w^2-4w+5\right)(w+6)=0$ (By inspection)

Comparing coefficients of w^2, $p=6-4=2$

Comparing coefficients of w, $q=-24+5=-19$

Method 2

Substitute $w=2-\mathrm{i}$ (or $w=2+\mathrm{i}$) into the given eqn,

$$(2-\mathrm{i})^3+p(2-\mathrm{i})^2+q(2-\mathrm{i})+30=0$$
$$(3-4\mathrm{i})(2-\mathrm{i})+p(3-4\mathrm{i})+q(2-\mathrm{i})+30=0$$
$$(6-3\mathrm{i}-8\mathrm{i}-4)+p(3-4\mathrm{i})+q(2-\mathrm{i})+30=0$$
$$(32+3p+2q)+(-11-4p-q)\mathrm{i}=0$$

Comparing the real parts, $3p+2q=-32$--- (1)

Comparing the imaginary parts, $4p+q=-11$---- (2)

(1) – (2) × 2: $3p-8p=-32+11\times 2$
$$-5p=-10$$
$$p=2$$

From (2): $q=-11-4\times 2=-19$

$\therefore p=2$, $q=-19$

(b)

Substitute $z=3+u\mathrm{i}$ into the given eqn,

$$(3+u\mathrm{i})^2+(-5+2\mathrm{i})(3+u\mathrm{i})+(21-\mathrm{i})=0$$
$$9+6u\mathrm{i}-u^2-15-5u\mathrm{i}+6\mathrm{i}-2u+21-\mathrm{i}=0$$
$$\left(15-2u-u^2\right)+(u+5)\mathrm{i}=0$$

Compare imaginary coefficient: $u+5=0$
$$u=-5$$

$\therefore z=3-5\mathrm{i}$

[Note: if using $15-2u-u^2=0$, need to reject $u=3$]

Method 1

Let the other root be w.

$z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=(z-3+5\mathrm{i})(z-w)$

Comparing coefficients of z,

$-5+2\mathrm{i}=-w-3+5\mathrm{i}$

$w=2+3\mathrm{i}$

Method 2

Let the other solution be $a+b\text{i}$,

$$z^2+(-5+2\text{i})z+(21-\text{i})$$
$$=(z-(3-5\text{i}))(z-(a+b\text{i}))$$
$$=z^2-(a+b\text{i})z-(3-5\text{i})z+(3-5\text{i})(a+b\text{i})$$
$$=z^2-\left[a+3+(b-5)\text{i}\right]z+(3-5\text{i})(a+b\text{i})$$

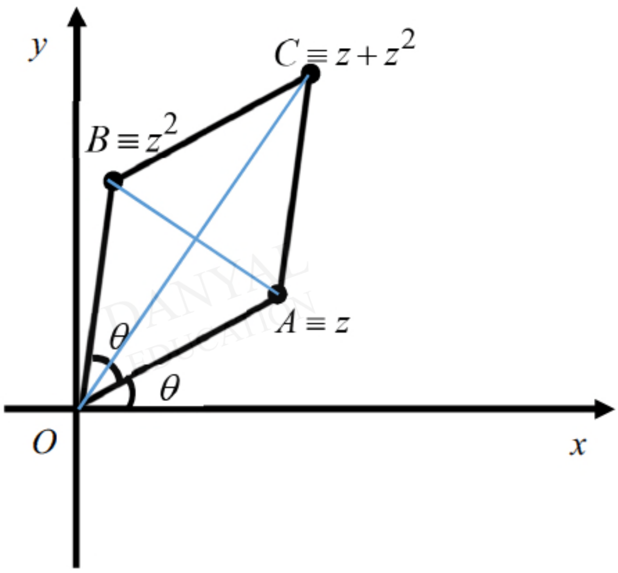
Compare the z term: $-(a+3)=-5 \quad \Rightarrow a=2$

$-(b-5)=2 \quad \Rightarrow b=3$

$\therefore z=2+3\text{i}$ is another root.

Q2

(i)



(ii)

Since *OACB* is a parallelogram with 4 equal sides, it is a **rhombus**.

(iii)

$$z+z^2$$
$$=\cos\theta+\mathrm{i}\sin\theta+(\cos\theta+\mathrm{i}\sin\theta)^2$$
$$=\cos\theta+\mathrm{i}\sin\theta+\cos^2\theta+2\mathrm{i}\cos\theta\sin\theta-\sin^2\theta$$
$$=(\cos\theta+\cos 2\theta)+\mathrm{i}(\sin\theta+\sin 2\theta)$$
$$=2\cos\frac{3\theta}{2}\cos\frac{\theta}{2}+2\mathrm{i}\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$$
$$=2\cos\frac{\theta}{2}\left[\cos\frac{3\theta}{2}+\mathrm{i}\sin\frac{3\theta}{2}\right]$$

Alternative

$$\arg(z+z^2)=\theta+\frac{\theta}{2}=\frac{3}{2}\theta$$
$$|z+z^2|=2OM=2\cos\left(\frac{\theta}{2}\right)$$
$$z+z^2=2\cos\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{3}{2}\theta\right)+\mathrm{i}\sin\left(\frac{3}{2}\theta\right)\right]$$
$$\therefore p=2,\ q=\frac{1}{2},\ k=\frac{3}{2}$$

Q3

(a)(i) Since $z=1+3\mathrm{i}$ is a root and the polynomial has real coefficients, $z=1-3\mathrm{i}$ is also a root to the polynomial.

Hence a quadratic factor of the polynomial is
$(z-(1+3\mathrm{i}))(z-(1-3\mathrm{i}))=\left(z^2-z(1+3\mathrm{i}+1-3\mathrm{i})+(1+3\mathrm{i})(1-3\mathrm{i})\right)=\left(z^2-2z+10\right)$

$z^4+2z^3+az^2+bz+50$
$=\left(z^2-2z+10\right)\left(Az^2+Bz+C\right)$ for some constants A, B and C.

By comparing coefficient of z^4 and z^3, $A=1$ and $B-2A=2 \Rightarrow B=4$
By comparing the constant term, $C=5$

Hence $z^4+2z^3+az^2+bz+50=\left(z^2-2z+10\right)\left(z^2+4z+5\right)$
Comparing coefficient of z^2 and z, we have $a=-8+10+5=7$ and $b=40-10=30$ (shown).

Solving $z^2+4z+5=0$, $z=\dfrac{-4\pm\sqrt{4^2-4(5)}}{2}=\dfrac{-4\pm\sqrt{-4}}{2}=\dfrac{-4\pm\sqrt{4}\sqrt{-1}}{2}=-2\pm\mathrm{i}$.
Hence the other roots are $z=1-3\mathrm{i}$, $z=-2+\mathrm{i}$ and $z=-2-\mathrm{i}$.

<u>Alternative Solution (more tedious):</u>
Since $1+3\mathrm{i}$ is a root,
$(1+3\mathrm{i})^4+2(1+3\mathrm{i})^3+a(1+3\mathrm{i})^2+b(1+3\mathrm{i})+50=0$--(1)

$(1+3\mathrm{i})^2=1^2+2(3\mathrm{i})+(3\mathrm{i})^2=(1-9)+6\mathrm{i}=-8+6\mathrm{i}$

$(1+3\mathrm{i})^3=(1+3\mathrm{i})(-8+6\mathrm{i})=(-8-18)+\mathrm{i}(6-24)=-26-18\mathrm{i}$

$(1+3\mathrm{i})^4=(-8+6\mathrm{i})^2=64-96\mathrm{i}-36=28-96\mathrm{i}$

Applying above results on (1),
$(28-96\mathrm{i})+2(-26-18\mathrm{i})+a(-8+6\mathrm{i})+b(1+3\mathrm{i})+50=0$
$(26-8a+b)+(-132+6a+3b)\mathrm{i}=0$
Comparing real and imaginary parts,
$26-8a+b=0$ and $-132+6a+3b=0$
equivalent to $-44+2a+b=0$
Solving, $-44-26+10a=0 \Rightarrow a=7$ and $b=8(7)-26=30$
$\therefore a=7,\ b=30$ (shown)

Since $z=1+3\mathrm{i}$ is a root and the polynomial has real coefficients, $z=1-3\mathrm{i}$ is also a root to the polynomial.

$$\begin{aligned}&z^4+2z^3+7z^2+30z+50\\&=\left(z-(1+3\mathrm{i})\right)\left(z-(1-3\mathrm{i})\right)\left(z^2+Az+B\right)\\&=\left(z^2-2z+10\right)\left(z^2+Az+B\right)\end{aligned}$$

By comparing coefficients, we have $A=4, B=5$.

Solving $z^2+4z+5=0$, $z=\dfrac{-4\pm\sqrt{-4}}{2}=\dfrac{-4\pm\sqrt{4}\sqrt{-1}}{2}=-2\pm\mathrm{i}$.

Hence the other roots are $z=1-3\mathrm{i}$, $z=-2+\mathrm{i}$ and $z=-2-\mathrm{i}$.

(a)(ii)Let $z=\mathrm{i}w$, then we get $(\mathrm{i}w)^4+2(\mathrm{i}w)^3+7(\mathrm{i}w)^2+30(\mathrm{i}w)+50=0$

$$\Rightarrow\ w^4-2\mathrm{i}w^3-7w^2+30\mathrm{i}w+50=0.$$

$z=\mathrm{i}w\ \Rightarrow\ w=-\mathrm{i}z.$

Hence the roots are $w=-\mathrm{i}-3$, $w=-\mathrm{i}+3$, $w=2\mathrm{i}+1$ and $w=2\mathrm{i}-1$.

(b) $|p|=|p^*|=\dfrac{\left|\left(-\frac{1}{\sqrt{3}}+\mathrm{i}\right)\right|^5}{\left|(1-\mathrm{i})\right|^4}=\dfrac{\left(\frac{2}{\sqrt{3}}\right)^5}{\left(\sqrt{2}\right)^4}=\dfrac{32}{4}\left(\dfrac{1}{\sqrt{3}}\right)^5=\dfrac{8}{9\sqrt{3}}$ or $\dfrac{8\sqrt{3}}{27}$

$$\begin{aligned}\arg(p)=-\arg(p^*)&=-\left(5\arg\left(-\frac{1}{\sqrt{3}}+\mathrm{i}\right)-4\arg(1-\mathrm{i})\right)+2\pi+2\pi\\&=-\left(5\left(\frac{2\pi}{3}\right)-4\left(-\frac{\pi}{4}\right)\right)+2\pi+2\pi\\&=-\frac{\pi}{3}\end{aligned}$$

$$p=\frac{8}{9\sqrt{3}}\left(\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)=\frac{8}{9\sqrt{3}}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)=\frac{4}{9\sqrt{3}}-\frac{4}{9}\mathrm{i}\text{ or }\frac{4\sqrt{3}}{27}-\frac{4}{9}\mathrm{i}$$