A Level H2 Math

Complex Numbers Test 5

Q1

The polynomial $P(z)$ has real coefficients. The equation $P(z)=0$ has a root $re^{i\theta}$, where $r>0$ and $0<\theta<\pi$. Write down a second root in terms of r and θ, and hence show that a quadratic factor of $P(z)$ is $z^2-2rz\cos\theta+r^2$. [2]

Let $P(z)=z^3+az^2+15z+18$ where a is a real number. One of the roots of the equation $P(z)=0$ is $3e^{i\left(\frac{2\pi}{3}\right)}$. By expressing $P(z)$ as a product of two factors with real coefficients, find a and the other roots of $P(z)=0$. [4]

Deduce the roots of the equation $18z^3+15z^2+az+1=0$. [2]

Q2

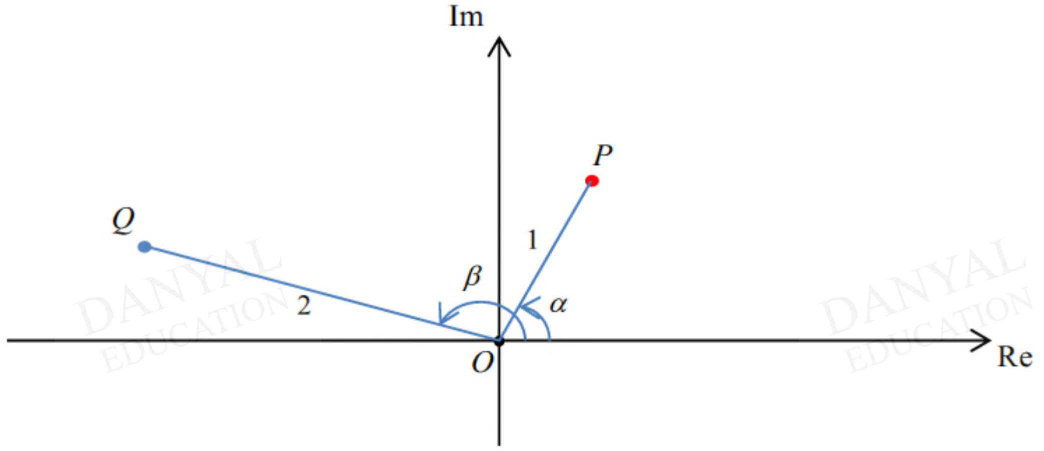
The complex numbers z and w satisfy the simultaneous equations

$$iz+w=2+i \quad \text{and} \quad 2w-(1+i)z=8+4i.$$

Find z and w in the form of $a+ib$, where a and b are real. [5]

Q3

For $\alpha, \beta \in \mathbb{R}$ such that $2\alpha < \beta$, the complex numbers $z_1 = e^{i\alpha}$ and $z_2 = 2e^{i\beta}$ are represented by the points P and Q respectively in the Argand diagram below.



Find the modulus and argument of the complex numbers given by $\frac{i}{2}z_2$ and $\frac{{z_1}^2}{z_2}$. [4]

Copy the given Argand diagram onto your answer script and indicate clearly the following points representing the corresponding complex numbers on your diagram.

(i) A: $\frac{i}{2}z_2$ [1]

(ii) B: $\frac{{z_1}^2}{z_2}$ [1]

You are expected to indicate clearly the relevant moduli and arguments for parts **(i)** and **(ii)** on your Argand diagram.

If $\beta = \frac{11}{12}\pi$, find the smallest positive integer n such that the point representing the complex number $(z_2)^n$ lies on the negative real axis. [3]

Answers

Complex Numbers Test 5

Q1

Second root is $re^{-i\theta}$.

Quadratic factor of P(z) is

$$\begin{aligned}&\left(z-re^{i\theta}\right)\left(z-re^{-i\theta}\right)\\&=z^2-\left(re^{i\theta}+re^{-i\theta}\right)z+\left(re^{i\theta}\right)\left(re^{-i\theta}\right)\\&=z^2-r\left(e^{i\theta}+e^{-i\theta}\right)z+r^2\\&=z^2-r\left(\cos\theta+i\sin\theta+\cos\theta-i\sin\theta\right)z+r^2\\&=z^2-\left(2r\cos\theta\right)z+r^2\end{aligned}$$

root of the equation is $3e^{i\left(\frac{2\pi}{3}\right)}$.

So $r=3$ and $\theta=\frac{2\pi}{3}$.

Quadratic factor is $z^2-2(3)\left(\cos\frac{2\pi}{3}\right)z+9 = z^2+3z+9$

hence $z^3+az^2+15z+18=(z^2+3z+9)(z+2)$

By comparing z^2 term, $a=5$

The roots of the equation $z^3+az^2+15z+18=0$ are

$3e^{i\left(\frac{2\pi}{3}\right)}$, $3e^{i\left(-\frac{2\pi}{3}\right)}$ and $-2=2e^{i(\pi)}$

$18z^3+15z^2+az+1=0$

$$z^3\left(18+15\left(\frac{1}{z}\right)+a\left(\frac{1}{z^2}\right)+\left(\frac{1}{z^3}\right)\right)=0$$

Since $z\neq 0$, and let $w=\frac{1}{z}$

We have $w^3+aw^2-2w+18=0$

Hence $w=3e^{i\left(\frac{2\pi}{3}\right)},3e^{i\left(-\frac{2\pi}{3}\right)},-2$

$\frac{1}{z}=3e^{i\left(\frac{2\pi}{3}\right)},3e^{i\left(-\frac{2\pi}{3}\right)},-2$

Since $\left|\frac{1}{z}\right|=\frac{1}{|z|}$ and $\arg\left(\frac{1}{z}\right)=-\arg(z)$

So $z=\frac{1}{3}e^{i\left(-\frac{2\pi}{3}\right)},\frac{1}{3}e^{i\left(\frac{2\pi}{3}\right)},-\frac{1}{2}$ are the roots of $18z^3+15z^2+az+1=0$

Q2

$iz + w = 2 + i ------(1)$

$2w - 1 - iz = \frac{20}{2 - i} ---(2)$

Let $w = 2 + i - iz ---(3)$

Substitute eq (3) into eq (2)

$2(2 + i - iz) - z - iz = 8 + 4i$

$4 + 2i - 3iz - z = 8 + 4i ----(5)$

Let $z = a + bi$

Substitute $z = a + bi$ into eq(5)

$4 + 2i - 3i(a + bi) - (a + bi) = 8 + 4i$

$4 + 2i - 3ai + 3b - a - bi = 8 + 4i$

Comparing real and imaginary parts:

$4 + 3b - a = 8 \text{(real parts)} ---(6)$

$2 - 3a - b = 4 \text{(imaginary parts)} ---(7)$

$Eq(6) \times 3 \; - \; eq(7)$

$10 + 10b = 20$

$10b = 10$

$b = 1$

Since $b = 1$, $4 + 3(1) - a = 8 \Rightarrow a = -1$

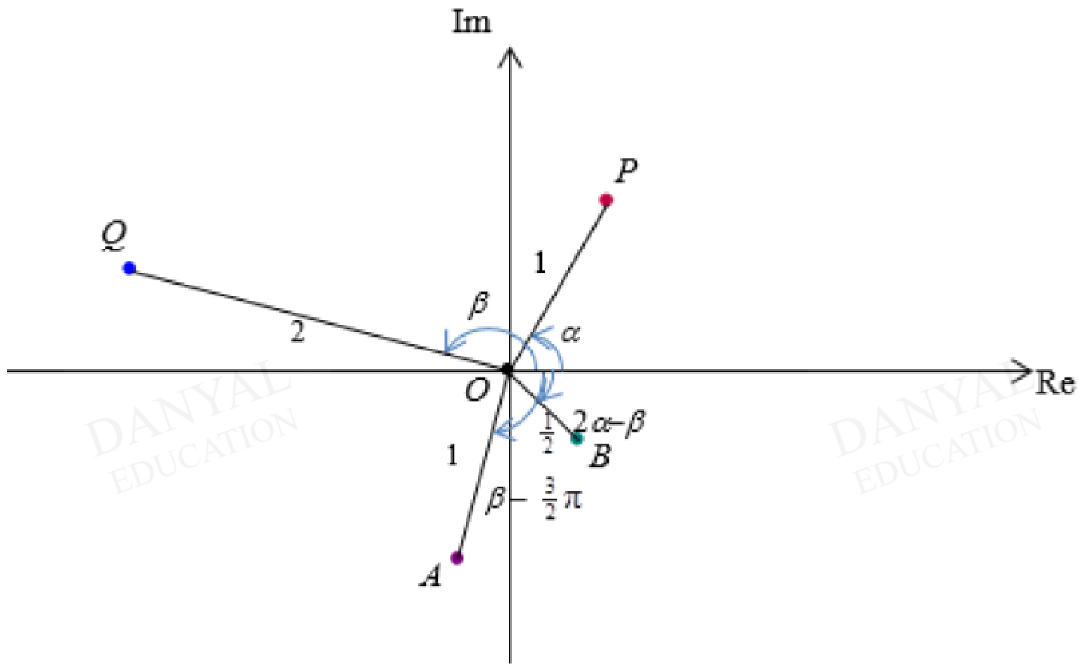
$\therefore z = -1 + i$

Substituting $z = -1 + i$ into eq(3) to solve for w

$w = 2 + i + i + 1 = 3 + 2i$

Answer: $z = -1 + i$ and $w = 3 + 2i$

Q3



$$\frac{i}{2}z_2=\left(\frac{1}{2}e^{i\frac{\pi}{2}}\right)\left(2e^{i\beta}\right)=e^{i\left(\beta+\frac{\pi}{2}\right)}$$

Modulus = 1

Argument = $\beta+\frac{\pi}{2}-2\pi=\beta-\frac{3\pi}{2}$

(i) Point A correctly plotted

$$\frac{z_1^{\ 2}}{z_2}=\frac{e^{i\alpha}e^{i\alpha}}{2e^{i\beta}}=\frac{1}{2}e^{i(2\alpha-\beta)}$$

Modulus = $\frac{1}{2}$

Argument = $2\alpha-\beta$

(ii) Point B correctly plotted

(b) $(z_2)^n=2^n e^{i\frac{11\pi}{12}n}$

Since the point lies on the negative real axis, $\arg(z_2)^n=(2k+1)\pi$ for $k\in\mathbb{Z}$.

$$\therefore\ \frac{11}{12}n\pi=(2k+1)\pi$$

$$\Rightarrow n=\frac{12}{11}(2k+1)$$

$\Rightarrow$ Smallest n required = 12