A Level H2 Math

Complex Numbers Test 4

Q1

The complex numbers z and w satisfy the following equations

$$2z + 3w = 20\ ,$$
$$w - zw^* = 6 + 22\text{i}\ .$$

(i) Find z and w in the form $a + b\text{i}$, where a and b are real, $a \neq 0$. [5]

(ii) Show z and w on a single Argand diagram, indicating clearly their modulus. State the relationship between z and w with reference to the origin O. [2]

Q2

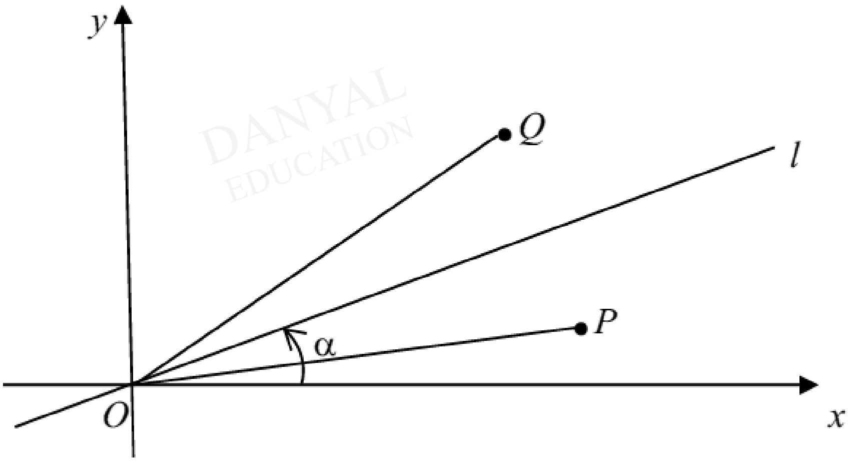
(a) If $u = 2 - \text{i}\sin^2\theta$ and $v = 2\cos^2\theta + \text{i}\sin^2\theta$ where $-\pi < \theta \leq \pi$, find $u - v$ in terms of $\sin^2\theta$, and hence determine the exact expression for $|u - v|$ and the exact value of $\arg(u - v)$. [6]

(b) The roots of the equation $x^2 + (\text{i} - 3)x + 2(1 - \text{i}) = 0$ are α and β, where α is a real number and β is not a real number. Find α and β. [4]

Q3

The diagram below shows the line l that passes through the origin and makes an angle α with the positive real axis, where $0 < \alpha < \frac{\pi}{2}$.

Point P represents the complex number z_1 where $0 < \arg z_1 < \alpha$ and length of OP is r units. Point P is reflected in line l to produce point Q, which represents the complex number z_2 .



Prove that $\arg z_1 + \arg z_2 = 2\alpha$. [2]

Deduce that $z_1 z_2 = r^2 \left(\cos 2\alpha + i \sin 2\alpha\right)$. [1]

Let R be the point that represents the complex number $z_1 z_2$. Given that $\alpha = \frac{\pi}{4}$, write down the cartesian equation of the locus of R as z_1 varies. [2]

Answers

Complex Numbers Test 4

Q1

(i)

$2z+3w=20 \quad ...(1)$

$w-zw^*=6+22\mathrm{i} \quad ...(2)$

From (1), $z=\dfrac{20-3w}{2}$

Substitute into (2),

$w-\left(\dfrac{20-3w}{2}\right)w^*=6+22\mathrm{i}$

$2w-(20-3w)w^*=12+44\mathrm{i}$

$2w-20w^*+3ww^*=12+44\mathrm{i}$

Let $w=a+b\mathrm{i}$

$2(a+b\mathrm{i})-20(a-b\mathrm{i})+3(a+b\mathrm{i})(a-b\mathrm{i})=12+44\mathrm{i}$

$2a+2b\mathrm{i}-20a+20b\mathrm{i}+3(\mathrm{a}^2+\mathrm{b}^2)=12+44\mathrm{i}$

$\left(3a^2-18a+3b^2\right)+\left(22b\right)\mathrm{i}=12+44\mathrm{i}$

Comparing real and imaginary parts,

$22b=44$

$\therefore\ b=2$

$3a^2-18a+3(2)^2=12$

$3a^2-18a+12=12$

$3a\left(a-6\right)=0$

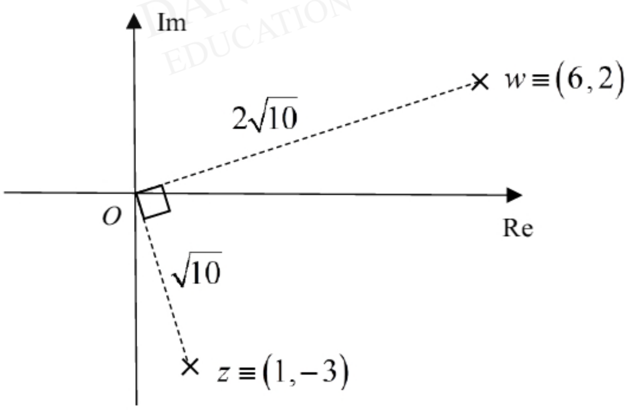
$a=0$ (rejected since $a\neq 0$), $a=6$

$\therefore w=6+2\mathrm{i}$

$z=\dfrac{20-3\left(6+2\mathrm{i}\right)}{2}$

$z=1-3\mathrm{i}$

(ii)



$\angle WOZ$ is $90°$

Q2

(a) $u = 2 - \mathrm{i}\sin^2\theta$, $v = 2\cos^2\theta + \mathrm{i}\sin^2\theta$

$$\begin{aligned}
u - v &= 2 - \mathrm{i}\sin^2\theta - 2\cos^2\theta - \mathrm{i}\sin^2\theta \\
&= 2 - 2\cos^2\theta - 2\mathrm{i}\sin^2\theta \\
&= 2\left(1-\cos^2\theta\right) - 2\mathrm{i}\sin^2\theta \\
&= \underline{\underline{2\sin^2\theta - 2\mathrm{i}\sin^2\theta}} \quad \text{or} \quad \underline{\underline{2\left(\sin^2\theta\right)(1-\mathrm{i})}}
\end{aligned}$$

$$\begin{aligned}
|u-v| &= 2\left|\sin^2\theta - \mathrm{i}\sin^2\theta\right| \\
&= 2\sqrt{\sin^4\theta + \sin^4\theta} \\
&= 2\sqrt{2\sin^4\theta} \\
&= \underline{\underline{2\sqrt{2}\sin^2\theta}}
\end{aligned}$$

or

$$\begin{aligned}
&2\left|\sin^2\theta\right|\left|1-\mathrm{i}\right| \\
&= 2\left(\sin^2\theta\right)\sqrt{1+1} \\
&= \underline{\underline{2\sqrt{2}\sin^2\theta}}
\end{aligned}$$

Note that $u - v$ lies in the 4th quadrant.

$$\begin{aligned}
\arg(u-v) &= -\tan^{-1}\frac{2\sin^2\theta}{2\sin^2\theta} \\
&= -\tan^{-1}1 = \underline{\underline{-\frac{\pi}{4}}}
\end{aligned}$$

Or:

$$\begin{aligned}
\arg(u-v) &= \arg\left(2\sin^2\theta - 2\mathrm{i}\sin^2\theta\right) = \arg\left[2\left(\sin^2\theta\right)(1-\mathrm{i})\right] \\
&= \arg\left(2\sin^2\theta\right) + \arg(1-\mathrm{i})
\end{aligned}$$

$$= 0+\left(-\frac{\pi}{4}\right) = -\frac{\pi}{4}$$

(b) Method 1 Solve α first then factorise quadratic expression or use sum of roots

$$x^2+(\mathrm{i}-3)x+2(1-\mathrm{i}) = 0$$

Sub. $x=\alpha\in\mathbb{R}$,

$$\alpha^2+(\mathrm{i}-3)\alpha+2(1-\mathrm{i}) = 0$$

$$(\alpha^2-3\alpha+2)+\mathrm{i}(\alpha-2) = 0$$

Comparing imaginary parts,

$$\alpha-2 = 0$$

$$\alpha = 2$$

$$x^2+(\mathrm{i}-3)x+2(1-\mathrm{i}) = (x-2)(x-\beta)$$

Comparing constants,

$$2(1-\mathrm{i}) = 2\beta$$

$$\therefore \beta = 1-\mathrm{i}$$

Or: Sum of roots, $\alpha+\beta = -(\mathrm{i}-3)$

$$2+\beta = 3-\mathrm{i}$$

$$\therefore \beta = 1-\mathrm{i}$$

Method 2 Factorise the quadratic expression first

$$x^2+(\mathrm{i}-3)x+2(1-\mathrm{i}) = (x-\alpha)(x-\beta)$$

Comparing coefficients of x,

$$\mathrm{i}-3 = -(\alpha+\beta)$$

$$\alpha+\beta = 3-\mathrm{i} \qquad (1)$$

Comparing constants,

$$\alpha\beta = 2-2\mathrm{i} \qquad (2)$$

From (1), $\beta = 3-\mathrm{i}-\alpha \qquad (3)$

Sub. (3) into (2), $\alpha(3-\mathrm{i}-\alpha) = 2-2\mathrm{i}$

$$3\alpha-\alpha^2-\alpha\mathrm{i} = 2-2\mathrm{i}$$

Comparing imaginary parts, $\alpha = 2$

Sub. into (3), $\beta = 3-\mathrm{i}-2$

$$\therefore \beta = 1-\mathrm{i}$$

Or:

Let $\beta=a+b\mathrm{i}$, where $a\in\mathbb{R}, b\in\mathbb{R}$ and $b\neq 0$

$$x^2+(\mathrm{i}-3)x+2(1-\mathrm{i}) = (x-\alpha)[x-(a+b\mathrm{i})]$$

Comparing coefficients of x,

$$\mathrm{i}-3 = -a-b\mathrm{i}-\alpha$$

$b = -1$ (Comparing imaginary parts)

$a+\alpha = 3 \quad (1)$ (Comparing real parts)

Comparing constants,

$$2-2\mathrm{i} = \alpha(a+b\mathrm{i})$$

$$= \alpha(a-\mathrm{i}) = \alpha a - \alpha\mathrm{i}$$

$$\underline{\underline{\alpha = 2}} \qquad \text{(Comparing imaginary parts)}$$

Sub. into (1), $a = 3-\alpha = 3-2 = 1$

$$\therefore \underline{\underline{\beta = 1-\mathrm{i}}}$$

Method 3 Solve x first using quadratic formula

$$x^2 + (\mathrm{i}-3)x + 2(1-\mathrm{i}) = 0$$

$$x = \frac{-(\mathrm{i}-3) \pm \sqrt{(\mathrm{i}-3)^2 - 4(1)[2(1-\mathrm{i})]}}{2}$$

$$= \frac{3-\mathrm{i} \pm \sqrt{\mathrm{i}^2 - 6\mathrm{i} + 9 - 8 + 8\mathrm{i}}}{2} = \frac{3-\mathrm{i} \pm \sqrt{2\mathrm{i}}}{2}$$

$$= \frac{3-\mathrm{i} \pm (1+\mathrm{i})}{2} \qquad \text{(use GC to find } \sqrt{2\mathrm{i}}\text{)}$$

$$= 2 \text{ or } 1-\mathrm{i}$$

$$\therefore \underline{\underline{\alpha = 2}} \text{ and } \underline{\underline{\beta = 1-\mathrm{i}}}$$

For comparison purpose:

If GC is **not** used to find $\sqrt{2\mathrm{i}}$, then the algebraic works will look as follows:

Let $\sqrt{2\mathrm{i}} = a + b\mathrm{i}$, where $a \in \mathbb{R}, b \in \mathbb{R}$

$$2\mathrm{i} = a^2 - b^2 + 2ab\mathrm{i}$$

Compring real parts, $a^2 - b^2 = 0$

$$a^2 = b^2$$

$$a = \pm b \quad (1)$$

Compring imaginary parts, $ab = 1 \quad (2)$

When $a = b$,

Sub. into (2), $a^2 = 1$

$$a = \pm 1$$

When $a = 1$, $b = 1$. When $a = -1, b = -1$

$$\pm\sqrt{2\mathrm{i}} = \pm(1+\mathrm{i})$$

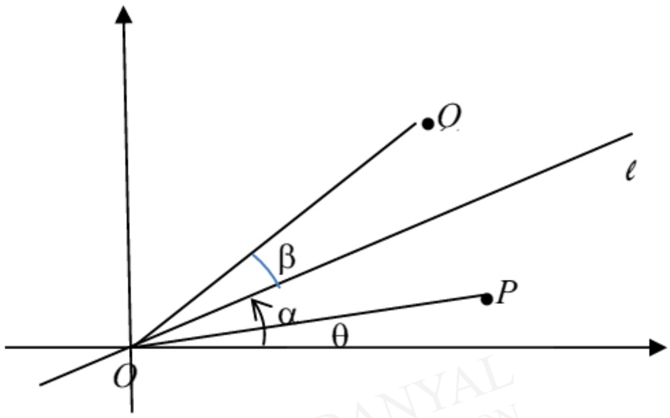
When $a = -b$

Sub. into (2), $-b^2 = 1 \quad (\text{NA } \because b \in \mathbb{R})$

$$\therefore x = \frac{3-\mathrm{i} \pm (1+\mathrm{i})}{2} = 2 \text{ or } 1-\mathrm{i}$$

$$\therefore \underline{\underline{\alpha = 2}} \text{ and } \underline{\underline{\beta = 1-\mathrm{i}}}$$

Q3



P ≡ $z_1 = re^{i\theta}$,
$|z_1| = r$ & arg (z_1) = θ

Let β be angle between lines OQ & *l*,
β = (α - θ) since line *l* **bisects** ∠POQ

arg z_1 + arg z_2
= θ+ (α + β)
= θ+ α + (α - θ)
= 2α

$|z_1 z_2| = |z_1||z_2| = r^2$ AND   arg (z_1 z_2) = arg z_1 + arg z_2 = 2α
Hence $z_1 z_2 = r^2\left(\cos 2\alpha + i\sin 2\alpha\right)$.

$\alpha = \frac{\pi}{4} \quad \Rightarrow \quad z_1 z_2 = r^2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = r^2 i$ (Purely imaginary).
Cartesian equation of the locus of *R* is $x = 0, y > 0$