**A Level H2 Math**

**Complex Numbers Test 3**

Q1

**Do not use a calculator in answering this question.**

**(a)** Showing your working clearly, find the complex numbers $z$ and $w$ which satisfy the simultaneous equations

$$iz + w = 2 \quad \text{and}$$

$$zw^* = 2 + 4i,$$

where $w^*$ is the complex conjugate of $w$. [5]

**(b)** The complex number $p$ is given by $a + ib$, where $a > 0$, $b < 0$, $a^2 + b^2 > 1$ and $\tan^{-1}\left(\frac{b}{a}\right) = -\frac{2\pi}{9}$.

**(i)** Express the complex number $\frac{1}{p^2}$ in the form $re^{i\theta}$, where $r$ is in terms of $a$ and $b$, and $-\pi < \theta \le \pi$. [2]

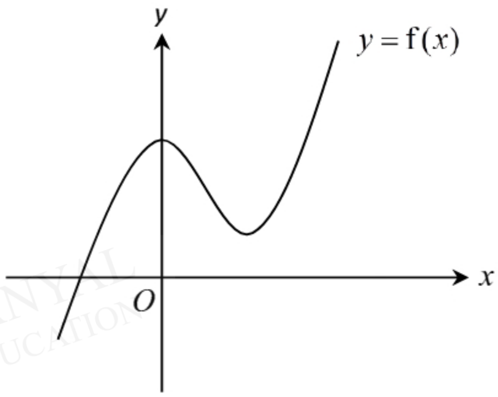
**(ii)** On a single Argand diagram, illustrate the points $P$ and $Q$ representing the complex numbers $p$ and $\frac{1}{p^2}$ respectively, labelling clearly their modulus and argument. [2]

**(iii)** It is given that $\angle OPQ = \alpha$. Using sine rule, show that $|p|^3 \approx \frac{\sqrt{3}}{2\alpha} - \frac{1}{2} - \frac{\alpha}{2\sqrt{3}}$ where $\alpha$ is small. [4]

Q2

**Do not use a graphic calculator in answering this question.**

**(a)**




It is given that $f(x)$ is a cubic polynomial with real coefficients. The diagram shows the curve with equation $y = f(x)$. What can be said about all the roots of the equation $f(x) = 0$? [2]

**(b)** The equation $2z^2 - (7+6i)z + 11 + ic = 0$, where $c$ is a non-zero real number, has a root $z = 3+4i$. Show that $c = -2$. Determine the other root of the equation in cartesian form. Hence find the roots of the equation $2w^2 + (-6+7i)w - 11 + 2i = 0$. [6]



**(c)** The complex number $z$ is given by $z = 1 + e^{i\alpha}$.

**(i)** Show that $z$ can be expressed as $2\cos\left(\frac{1}{2}\alpha\right)e^{i\left(\frac{1}{2}\alpha\right)}$. [2]

**(ii)** Given $\alpha = \frac{1}{3}\pi$ and $w = -1 - \sqrt{3}i$, find the exact modulus and argument of $\left(\frac{z}{w^3}\right)^*$. [5]

Q3

The complex number $z$ is given by $z = re^{i\theta}$, where $r > 0$ and $0 \le \theta \le \pi$. It is given that the complex number $w = \left(-\sqrt{3} - i\right)z$.

**(i)** Find $|w|$ in terms of $r$, and $\arg w$ in terms of $\theta$. [2]

**(ii)** Given that $\frac{z^8}{w^*}$ is purely imaginary, find the three smallest values of $\theta$ in terms of $\pi$. [5]

**Answers**

**Complex Numbers Test 3**

Q1

**5** **(a)**

$iz + w = 2$ ------(1)

$zw^* = 2 + 4i$ ------(2)

From (1),

$z = \frac{2-w}{i} = -i(2-w)$ ------(3)

Substitute (3) into (2) and let $w = x + iy$:

$-i(2-w)w^* = 2+4i$

$-i(2w^* - ww^*) = 2+4i$

$-i\left[2(x-iy)-(x^2+y^2)\right] = 2+4i$

$-2y - i(2x - x^2 - y^2) = 2+4i$

Comparing real and imaginary parts,

$-2y = 2 \Rightarrow y = -1$

$-2x + x^2 + y^2 = 4$

$\Rightarrow -2x + x^2 + (-1)^2 = 4$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0$

$\Rightarrow x = 3$ or $x = -1$

$\therefore w = 3 - i$ or $w = -1 - i$

If $w = 3 - i$, $z = -i(2-(3-i)) = 1 + i$.

If $w = -1 - i$, $z = -i(2-(-1-i)) = 1 - 3i$.

**(b)(i)**

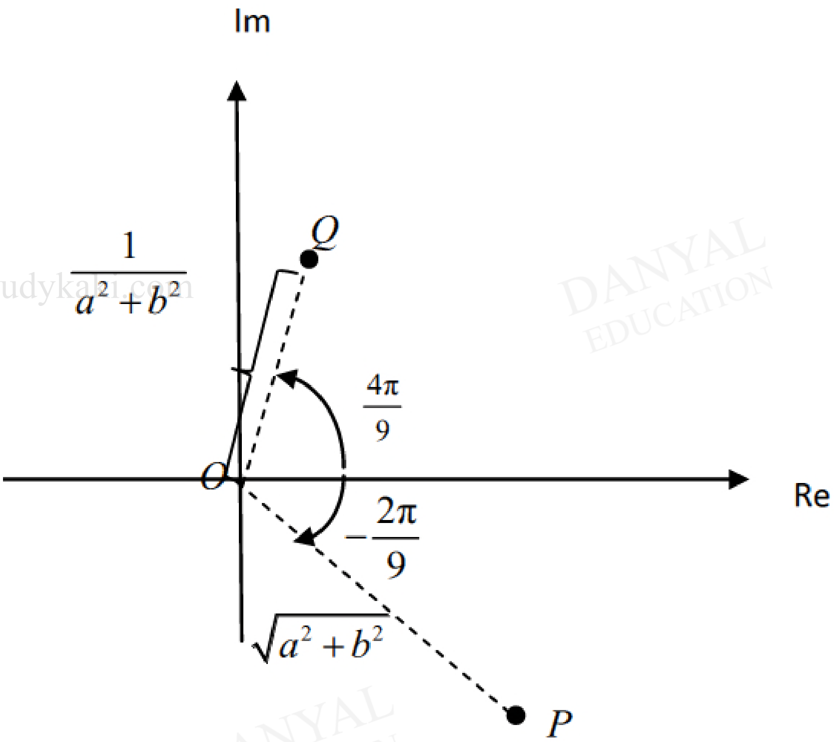
$$\left|\frac{1}{p^2}\right| = \frac{1}{|p|^2} = \frac{1}{\left(\sqrt{a^2+b^2}\right)^2} = \frac{1}{a^2+b^2}$$

$$\arg\left(\frac{1}{p^2}\right) = -2\arg(p) = -2\left(\frac{-2\pi}{9}\right) = \frac{4\pi}{9}$$

$$\therefore \frac{1}{p^2} = \frac{1}{a^2+b^2}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{9}\right)}$$



**(b)(ii)**






**(b)(iii)**

Given $\angle OPQ = \alpha$, $\dfrac{\sin\alpha}{1/\left(a^2+b^2\right)} = \dfrac{\sin\left(\dfrac{\pi}{3}-\alpha\right)}{\sqrt{a^2+b^2}}$

$$
\begin{aligned}
\left(a^2+b^2\right)^{\frac{3}{2}} &= \frac{\sqrt{3}\cos\alpha-\sin\alpha}{2\sin\alpha} \\
&\approx \frac{\sqrt{3}\left(1-\frac{x^2}{2}\right)-\left(x-\frac{x^3}{6}\right)}{2\left(x-\frac{x^3}{6}\right)} \\
&= \frac{1}{2}\left[\frac{1}{x}\left(\sqrt{3}-x-\frac{\sqrt{3}x^2}{2}+\frac{x^3}{6}\right)\left(1-\frac{x^2}{6}\right)^{-1}\right] \\
&\approx \frac{1}{2}\left[\frac{1}{x}\left(\sqrt{3}-x-\frac{\sqrt{3}x^2}{2}+\frac{x^3}{6}\right)\left(1+(-1)\left(-\frac{x^2}{6}\right)\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{\sqrt{3}}{x}-1-\frac{\sqrt{3}x}{2}+\frac{x^2}{6}\right)\left(1+\frac{x^2}{6}\right)\right] \\
&= \frac{1}{2}\left[\frac{\sqrt{3}}{x}+\frac{\sqrt{3}}{6}x-1-\frac{x^2}{6}-\frac{\sqrt{3}x}{2}+\frac{x^2}{6}\right] \\
&= \frac{1}{2}\left[\frac{\sqrt{3}}{x}-\frac{2\sqrt{3}}{6}x-1\right] \\
&= \frac{\sqrt{3}}{2x}-\frac{1}{2}-\frac{1}{2\sqrt{3}}x
\end{aligned}
$$

Q2

(a) Since the curve shows only one $x$-intercept, it means that there is only one real root in the equation $f(x)=0$.

Since the equation has all real coefficients, then the two other roots must be non-real and they are conjugate pair.

(b) Since $z=3+4i$ is a root of $2z^2-(7+6i)z+11+ic=0$,

$2(3+4i)^2-(7+6i)(3+4i)+11+ic=0$

$2(9+24i-16)-(21+28i+18i-24)+11+ic=0$

Comparing the Im - part,

$2+c=0$

$\therefore c=-2$ (shown)

Since $z=3+4i$ is a root of $2z^2-(7+6i)z+11-2i=0$,

$2z^2-(7+6i)z+11-2i=\left[z-(3+4i)\right](2z-a)$, where $a\in\mathbb{C}$

Comparing the coefficient of constant term,

$11-2i=a(3+4i)$

$a=\frac{11-2i}{3+4i}=\frac{(11-2i)(3-4i)}{25}=\frac{25-50i}{25}=1-2i$

$2z-(1-2i)=0\Rightarrow z=\frac{1}{2}-i$

Therefore, the other root is $\frac{1}{2}-i$.

Replace $z$ by $\mathrm{i}w$

$$2(\mathrm{i}w)^2-(7+6\mathrm{i})(\mathrm{i}w)+11-2\mathrm{i}=0$$

$$-2w^2-(-6+7\mathrm{i})w+11-2\mathrm{i}=0$$

$$2w^2+(-6+7\mathrm{i})w-11+2\mathrm{i}=0$$

$$\mathrm{i}w=3+4\mathrm{i} \Rightarrow w=4-3\mathrm{i} \quad \text{or} \quad \mathrm{i}w=\frac{1}{2}-\mathrm{i} \Rightarrow w=-1-\frac{1}{2}\mathrm{i}$$

$\therefore$ The roots of the equation are $4-3\mathrm{i}$ and $-1-\frac{1}{2}\mathrm{i}$.

<u>Alternative Method:</u>

$$2z^2-(7+6\mathrm{i})z+11-2\mathrm{i}=0$$

Let the other root be $a+b\mathrm{i}$.

Sum of the roots $=3+4\mathrm{i}+a+b\mathrm{i}=\frac{7+6\mathrm{i}}{2}=\frac{7}{2}+3\mathrm{i}$

Comparing real and imaginary parts:

$$a+3=\frac{7}{2}\Rightarrow a=\frac{1}{2}$$

$$4+b=3\Rightarrow b=-1$$

The other root is $\frac{1}{2}-\mathrm{i}$

(c) (i) $z=1+\mathrm{e}^{\mathrm{i}\alpha}$

$$=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}(\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}+\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}})$$

$$=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\left[2\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\right)\right]$$

$$=2\cos\frac{\alpha}{2}\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}} \text{ (shown)}$$

<u>Alternative Method:</u>

$$z=1+\mathrm{e}^{\mathrm{i}\alpha}$$

$$=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}(\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}+\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}})$$

$$=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\left(\cos\left(-\frac{\alpha}{2}\right)+\mathrm{i}\sin\left(-\frac{\alpha}{2}\right)+\cos\frac{\alpha}{2}+\mathrm{i}\sin\frac{\alpha}{2}\right)$$

$$=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\left[\cos\frac{\alpha}{2}-\mathrm{i}\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}+\mathrm{i}\sin\frac{\alpha}{2}\right]$$

$$=2\cos\frac{\alpha}{2}\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}} \text{ (shown)}$$

Alternative Method:

$$z = 1 + e^{i\alpha}$$
$$= 1 + \cos\alpha + i\sin\alpha$$
$$= 1 + 2\cos^2\frac{\alpha}{2} - 1 + i\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)$$
$$= 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$
$$= 2\cos\frac{\alpha}{2}e^{i\frac{\alpha}{2}} \text{ (shown)}$$

(ii) $$\left|\left(\frac{z}{w^3}\right)^*\right| = \left|\left(\frac{z}{w^3}\right)\right| = \frac{|z|}{|w|^3} = \frac{\left|2\cos\frac{\pi}{6}\right|}{\left(\sqrt{1+3}\right)^3} = \frac{2\left(\frac{\sqrt{3}}{2}\right)}{(2)^3} = \frac{\sqrt{3}}{8}$$

$$\arg\left(\frac{z}{w^3}\right)^* = -\arg\left(\frac{z}{w^3}\right) = -\left[\arg(z) - 3\arg(w)\right]$$
$$= -\frac{\alpha}{2} + 3\left(-\frac{2\pi}{3}\right) = -\frac{\pi}{6} - 2\pi$$
$$\therefore \arg\left(\frac{z}{w^3}\right)^* = -\frac{\pi}{6}$$

Q3

**(i)**

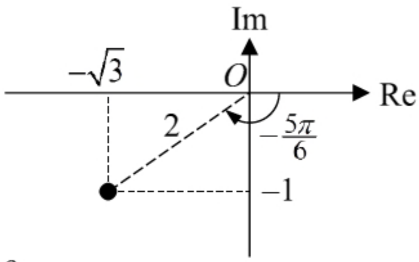
Method 1

$w=\left(-\sqrt{3}-\mathrm{i}\right)z$

$=[2\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}]r\,\mathrm{e}^{\mathrm{i}\theta}$

$=2r\,\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}+\theta\right)}$

$\therefore |w|=2r$, $\arg w=-\dfrac{5\pi}{6}+\theta$




Method 2

$|w|=\left|\left(-\sqrt{3}-\mathrm{i}\right)z\right|$

$=\left|\left(-\sqrt{3}-\mathrm{i}\right)\right||z|$

$=2r$

$\arg w=\arg\left(\left[-\sqrt{3}-\mathrm{i}\right]z\right)$

$=\arg\left(-\sqrt{3}-\mathrm{i}\right)+\arg z$

$=-\dfrac{5\pi}{6}+\theta$

**(ii)**

Method 1

$\arg\left(\dfrac{z^8}{w^*}\right)=\arg(z^8)-\arg(w^*)$ ← $\arg(w^*)=-\arg(w)$

$=8\theta+\arg w$

$=8\theta+\left(-\dfrac{5\pi}{6}+\theta\right)$ ← From **(ii)**

$=9\theta-\dfrac{5\pi}{6}$

For $\dfrac{z^8}{w^*}$ to be purely imaginary,

$\arg\left(\dfrac{z^8}{w^*}\right)=\ldots,-\dfrac{3\pi}{2},\ -\dfrac{\pi}{2},\ \dfrac{\pi}{2},\ \dfrac{3\pi}{2},\ \ldots$

$\therefore\ 9\theta-\dfrac{5\pi}{6}=\ldots,-\dfrac{3\pi}{2},\ -\dfrac{\pi}{2},\ \dfrac{\pi}{2},\ \dfrac{3\pi}{2},\ \ldots$

$9\theta=\ldots,-\dfrac{2\pi}{3},\ \dfrac{\pi}{3},\ \dfrac{4\pi}{3},\ \dfrac{7\pi}{3},\ \ldots$

$\theta=\ldots,-\dfrac{2\pi}{27},\ \dfrac{\pi}{27},\ \dfrac{4\pi}{27},\ \dfrac{7\pi}{27},\ \ldots$

$\therefore$ the three smallest values of $\theta$ are $\dfrac{\pi}{27}$, $\dfrac{4\pi}{27}$ and $\dfrac{7\pi}{27}$.

Method 2

$$\frac{z^8}{w^*} = \frac{(r\,\mathrm{e}^{\mathrm{i}\theta})^8}{2r\,\mathrm{e}^{\mathrm{i}\left[-\left(-\frac{5\pi}{6}+\theta\right)\right]}} = \frac{r^8\,\mathrm{e}^{\mathrm{i}(8\theta)}}{2r\,\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{6}-\theta\right)}}$$

$$= \frac{r^7}{2}\mathrm{e}^{\mathrm{i}\left[8\theta-\left(\frac{5\pi}{6}-\theta\right)\right]}$$

$$= \frac{r^7}{2}\mathrm{e}^{\mathrm{i}\left(9\theta-\frac{5\pi}{6}\right)}$$

For $\frac{z^8}{w^*}$ to be purely imaginary,

$\arg\left(\frac{z^8}{w^*}\right) = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$

$$\therefore\ 9\theta - \frac{5\pi}{6} = \frac{\pi}{2} + k\pi$$

$$9\theta = \frac{4\pi}{3} + k\pi$$

$$\theta = \frac{4\pi}{27} + \frac{k\pi}{9}$$



When $k = -2$, $\theta = -\frac{2\pi}{27}$

When $k = -1$, $\theta = \frac{\pi}{27}$

When $k = 0$, $\theta = \frac{4\pi}{27}$

When $k = 1$, $\theta = \frac{7\pi}{27}$

$\therefore$ the three smallest values of $\theta$ are $\frac{\pi}{27}$ , $\frac{4\pi}{27}$ and $\frac{7\pi}{27}$ .