A Level H2 Math

Complex Numbers Test 2

Q1

The cubic equation $az^3 - 31z^2 + 212z + b = 0$, where a and b are real numbers, has a complex root $z = 1 - 3i$.

(i) Explain why the equation must have a real root. [2]

(ii) Find the values of a and b and the real root, showing your working clearly. [5]

Q2

Do not use a calculator in answering this question.

(a) Solve the simultaneous equations

$$z - 4w = 11 + 6i \text{ and } 3z + 6iw = 27$$

giving z and w in the form $x + iy$ where x and y are real. [4]

(b) **(i)** The complex numbers z and w are given as $z = 4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$ and $w = 1 + i\sqrt{3}$. w^* denotes the conjugate of w. Find the modulus r and the argument θ of $\frac{w^*}{z^2}$, where $r > 0$ and $-\pi < \theta \le \pi$. [3]

(ii) Find the set of possible values of n such that $\left(\frac{w^*}{z^2}\right)^n$ is purely imaginary. [3]

Q3

(a) The complex numbers z and w satisfy the simultaneous equations

$$z + w^* + 5\text{i} = 10 \quad \text{and} \quad |w|^2 = z + 18 + \text{i}.$$

Find z and w. [4]

(b) **(i)** It is given that $2+\text{i}$ is a root of the equation $z^2 - 5z + 7 + \text{i} = 0$. Find the second root of the equation in cartesian form, showing your working clearly. [2]

(ii) Hence find the roots of the equation $-\text{i}w^2 + 5w + 7\text{i} - 1 = 0$. [2]

(c) The complex number z is given by $z = -a + a\text{i}$, where a is a positive real number.

(i) It is given that $w = -\dfrac{\sqrt{2}z^*}{z^4}$. Express w in the form $r\text{e}^{\text{i}\theta}$, in terms of a, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

(ii) Find the two smallest positive whole number values of n such that $\text{Re}\,(w^n) = 0$. [3]

Answers

Complex Numbers Test 2

Q1

(i)	Since the **coefficients** of $az^3 - 31z^2 + 212z + b = 0$ are **all real**, **complex roots occur in conjugate pair**. Since a **cubic equation has three roots**, the third root must be a real root.
(ii)	Since $1-3\text{i}$ is a root of $az^3 - 31z^2 + 212z + b = 0$, $a(1-3\text{i})^3 - 31(1-3\text{i})^2 + 212(1-3\text{i}) + b = 0$ $a(-26+18\text{i}) - 31(-8-6\text{i}) + 212(1-3\text{i}) + b = 0$ $(-26a+460+b)+(18a-450)\text{i} = 0$ Comparing real and imaginary parts: $-26a+460+b=0$ ----------- (1) $18a-450=0$ -----------------(2) From (2), $a=25$, $b=190$ $(z-(1-3\text{i}))(z-(1+3\text{i}))$ $=z^2-2z+10$ $25z^3-31z^2+212z+190=(z^2-2z+10)(cz+d)$ Comparing coefficient of z^3: $c=25$ Comparing constant: $190=10d$ $d=19$ The real root is $-\frac{19}{25}$.

Q2

(a)

$z-4w=11+6\text{i}$

$z=4w+11+6\text{i}$

Sub above equation into $3z+6iw=27$,

$3(4w+11+6\text{i})+6\text{i}w=27$

$12w+33+18\text{i}+6\text{i}w=27$

$w(12+6\text{i})=-6-18\text{i}$

$w=\dfrac{-6-18\text{i}}{12+6\text{i}}$

$=-1-\text{i}$

$z=4w+11+6\text{i}$

$=4(-1-\text{i})+11+6\text{i}$

$=7+2\text{i}$

ALT

$z-4w=11+6\text{i}$

$\times 3,\quad 3z-12w=33+18\text{i}\ \ldots(1)$

$3z+6\text{i}w=27\ \ldots(2)$

(2) − (1),

$6\text{i}w+12w=-6-18\text{i}$

$w=\dfrac{-6-18\text{i}}{12+6\text{i}}$

$=-1-\text{i}$

$z=4w+11+6\text{i}$

$=4(-1-\text{i})+11+6\text{i}$

$=7+2\text{i}$

(bi)

$|z| = 4$ & $\arg z = -\dfrac{\pi}{3}$

$|w| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$ & $\arg w = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$

$$\left|\frac{w^*}{z^2}\right| = \frac{|w^*|}{|z^2|} = \frac{|w|}{|z|^2} = \frac{2}{16} = \frac{1}{8}$$

$$\begin{aligned}\arg\left(\frac{w^*}{z^2}\right) &= \arg\left(w^*\right) - \arg\left(z^2\right) \\ &= -\arg w - 2\arg z \\ &= -\left(\frac{\pi}{3}\right) - 2\left(-\frac{\pi}{3}\right) \\ &= \frac{\pi}{3}\end{aligned}$$

(bii)

$$\frac{w^*}{z^2} = \frac{1}{8}\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

$$\left(\frac{w^*}{z^2}\right)^n = \left(\frac{1}{8}\right)\left[\cos\left(\frac{n\pi}{3}\right) + i\sin\left(\frac{n\pi}{3}\right)\right]$$

For $\left(\dfrac{w^*}{z^2}\right)^n$ to be purely imaginary,

$$\cos\left(\frac{n\pi}{3}\right) = 0$$

$$\begin{aligned}\frac{n\pi}{3} &= \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots \\ &= \frac{(2m+1)\pi}{2}, m \in \mathbb{Z} \\ n &= \frac{3(2m+1)}{2}, m \in \mathbb{Z}\end{aligned}$$

$$\left\{n : n = \frac{3(2m+1)}{2},\ m \in \mathbb{Z}\right\}$$

Q3

(a)

$z = 10 - w^* - 5\mathrm{i}$

$|w|^2 = 10 - w^* - 5\mathrm{i} + 18 + \mathrm{i}$

$|w|^2 + w^* = 28 - 4\mathrm{i}$

Let $w = a + b\mathrm{i}$,

$a^2 + b^2 + a - b\mathrm{i} = 28 - 4\mathrm{i}$

By comparing, $b = 4$,

$a^2 + (4)^2 + a = 28$

$a^2 + a - 12 = 0$

$(a+4)(a-3) = 0$

$a = -4 \quad \text{or} \quad a = 3$

$\therefore w = 3 + 4\mathrm{i} \quad \text{or} \quad w = -4 + 4\mathrm{i}$

When $w = 3 + 4\mathrm{i}$,

$z = 10 - (3 - 4\mathrm{i}) - 5\mathrm{i}$

$\quad = 7 - \mathrm{i}$

When $w = -4 + 4\mathrm{i}$,

$z = 10 - (-4 + 4\mathrm{i}) - 5\mathrm{i}$

$\quad = 14 - 9\mathrm{i}$

Most students were able to do this question except for the occasional slips in algebraic manipulation.

A number of students mistook $|w|^2$ for w^2.

Presentation for simultaneous equation is unclear.

(b)(i)

$z^2 - 5z + 7 + \mathrm{i} = [z - (2+\mathrm{i})][z - k]$

By comparing coefficient of *z*:

$z^2 - 5z + 7 + \mathrm{i} = [z - (2+\mathrm{i})][z - k]$

$-5 = -k - (2 + \mathrm{i})$

$k = 3 - \mathrm{i}$

The second root is $3 - \mathrm{i}$

Generally well done.

(ii)

$-\mathrm{i}w^2 + 5w + 7\mathrm{i} - 1 = 0$

$-w^2 - 5\mathrm{i}w + 7 + \mathrm{i} = 0$

$(\mathrm{i}w)^2 - 5(\mathrm{i}w) + 7 + \mathrm{i} = 0$

$\mathrm{i}w = 2 + \mathrm{i} \quad \text{or} \quad \mathrm{i}w = 3 - \mathrm{i}$

$w = 1 - 2\mathrm{i} \quad \text{or} \quad w = -1 - 3\mathrm{i}$

Badly done. Most students fail to identify the term to replace.

(c)(i)

<u>Method ①:</u>

$z = -a + a\mathrm{i}$

$\quad = a\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$

$$w = \left(\mathrm{e}^{\mathrm{i}(\pi)}\right)\frac{\sqrt{2}a\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}}{4a^4\mathrm{e}^{\mathrm{i}(3\pi)}}$$

$$= \frac{1}{2a^3}\mathrm{e}^{\mathrm{i}\left(-\frac{11\pi}{4}+2\pi\right)}$$

$$= \frac{1}{2a^3}\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}$$

Badly done.

Most students prefer to simplify the denominator but had problems with the algebraic manipulation.

Most students got the argument wrong as they left their answer as $-\frac{1}{2a^3}\mathrm{e}^{\mathrm{i}\alpha}$, or they mistook $\arg(-\sqrt{2}z^*) = -\sqrt{2}\arg(z^*)$.

Another common mistake was that many students left the argument of z as $\frac{\pi}{4}$.

(ii)

If $\mathrm{Re}\,(w^n) = 0$,

$$n\left(-\frac{3\pi}{4}\right) = \frac{\pi}{2} + k\pi, \text{ where } k \in \mathbb{Z}$$

$$n = -\frac{2}{3}(1 + 2k), \text{ where } k \in \mathbb{Z}$$

Three smallest positive whole number values of *n* are 2, 6.

Most students got the method marks but lost the last mark as their argument was wrong.