A Level H2 Math

Complex Numbers Test 1

Q1

(a) Given that $2z+1=|w|$ and $2w-z=4+8\mathrm{i}$, solve for w and z. [5]

(b) Find the exact values of x and y, where $x, y \in \mathbb{R}$, such that $2\mathrm{e}^{-\left(\frac{3+x+\mathrm{i}y}{\mathrm{i}}\right)}=1-\mathrm{i}$. [4]

Q2

Given that $1+\mathrm{i}$ is a root of the equation $z^3-4(1+\mathrm{i})z^2+(-2+9\mathrm{i})z+5-\mathrm{i}=0$, find the other roots of the equation. [4]

Q3

It is given that $z=-1-\mathrm{i}\sqrt{3}$.

(i) Given that $\dfrac{(\mathrm{i}z)^n}{z^2}$ is purely imaginary, find the smallest positive integer n. [4]

The complex number w is such that $|wz|=4$ and $\arg\left(\dfrac{w^*}{z^2}\right)=-\dfrac{5\pi}{6}$.

(ii) Find the value of $|w|$ and the exact value of $\arg(w)$ in terms of π. [3]

On an Argand diagram, points A and B represent the complex numbers w and z respectively.

(iii) Referred to the origin O, find the exact value of the angle OAB in terms of π. Hence, or otherwise, find the exact value of $\arg(z-w)$ in terms of π. [2]

Answers

Complex Numbers Test 1

Q1

(a)

$2z+1=|w|$ ....(1)

$2w-z=4+8\text{i}$.......(2)

$2z+1=$ a positive real number

$\Rightarrow$ Let $z=x$ and $w=a+b\text{i}$

From (2): $2(a+b\text{i})-x=4+8\text{i}$

$\Rightarrow$ Comparing Re and Im parts,

$2a-x=4$

$2b=8\Rightarrow b=4$

From (1): $2x+1=\sqrt{a^2+b^2}$....(3)

Substitute $b=4$ and $x=2a-4$ into (3):

$2(2a-4)+1=\sqrt{a^2+16}\Rightarrow(4a-7)^2=a^2+16$

$16a^2-56a+49=a^2+16\Rightarrow 15a^2-56a+33=0$

$\Rightarrow a=\frac{11}{15}$ or $a=3$

$\Rightarrow x=-\frac{98}{15}$ or $x=2$

but $2z+1=$ a positive real number

$\Rightarrow$ when $x=-\frac{98}{15}$, $2z+1=2\left(-\frac{98}{15}\right)+1<0$

$\Rightarrow$ reject $x=-\frac{98}{15}$ and $a=\frac{11}{15}$

$\Rightarrow x=2,\ a=3, b=4$

$\Rightarrow z=2,\ w=3+4\text{i}$

Many students failed to see that z is a real number from eqn (1), resulting in solving simultaneous eqns with many unknown, which most failed to simplify and continue to solve correctly.

Some common mistakes:

1. $|w|=w$
2. $|w|=\pm w$
3. $|w|=\sqrt{a^2+(ib)^2}=\sqrt{a^2-b^2}$

(b)

$2\text{e}^{-\left(\frac{3+x+\text{i}y}{\text{i}}\right)}=1-\text{i}$

$2\text{e}^{3\text{i}+x\text{i}-y}=\sqrt{2}\text{e}^{\text{i}\left(-\frac{\pi}{4}\right)}$

$2\text{e}^{-y}\text{e}^{\text{i}(3+x)}=\sqrt{2}\text{e}^{\text{i}\left(-\frac{\pi}{4}\right)}$

$\Rightarrow$ By comparing modulus and args:

$2\text{e}^{-y}=\sqrt{2}$ and $\qquad 3+x=-\frac{\pi}{4}$

$-y=\ln\left(\frac{\sqrt{2}}{2}\right) \qquad \Rightarrow x=-\frac{\pi}{4}-3$

$\Rightarrow y=-\ln\left(\frac{\sqrt{2}}{2}\right)$ (or $\ln\sqrt{2}$ or $\frac{1}{2}\ln 2$)

It's a surprise to see that many students didn't write $1-\text{i}$ in $r\text{e}^{\text{i}\theta}$ form to solve the problem. Even if some did it, they made a mistake in the value of $\theta=\frac{3}{4}\pi$ or $\frac{1}{4}\pi$.

In general, students have good idea how to manipulate $-\left(\frac{3+x+\text{i}y}{\text{i}}\right)$ to get $-y+3\text{i}+x\text{i}$ and they also have clear idea of comparing the modulus and argument terms.

Q2

$z^3-4(1+\text{i})z^2+(-2+9\text{i})z+5-\text{i}=0$

$\left(z-(1+\text{i})\right)\left(Az^2+Bz+C\right)=0$

By comparing coefficients,

$z^3: A=1$

$z^0: -(1+\text{i})C=5-\text{i}$

$\Rightarrow C=\dfrac{5-\text{i}}{-(1+\text{i})}=-2+3\text{i}$

$z^2: B-(1+\text{i})=-4(1+\text{i})$

$\Rightarrow B=-3(1+\text{i})$

$\Rightarrow \left(z-(1+\text{i})\right)\left(z^2-3(1+\text{i})z-2+3\text{i}\right)=0$

Solving $\left(z^2-3(1+\text{i})z-2+3\text{i}\right)=0$:

$$z=\frac{-\left(-3(1+\text{i})\right)\pm\sqrt{\left(-3(1+\text{i})\right)^2-4(1)\left(-2+3\text{i}\right)}}{2(1)}$$

$$=\frac{3+3\text{i}\pm\sqrt{8+6\text{i}}}{2}$$

$$=\frac{3+3\text{i}\pm(3+\text{i})}{2}=3+2\text{i} \text{ or } \text{i}$$

$\therefore$ other 2 roots are $z=3+2\text{i}$ or $z=\text{i}$

Quite a large number of students say that $1-\text{i}$ is another root, which is wrong because not all the coefficients are real. Students who did this gets a 0.

When comparing coefficients, many students use $a+ib, c+\text{i}d$ as the two other roots which resulted in unnecessarily tedious and complicated working.

About half who used the quadratic formula had problem evaluating $\sqrt{8+6\text{i}}$, which can be done using GC.

Q3

(i)

$|z| = \sqrt{1^2 + \sqrt{3}^2} = 2 \qquad \arg z = -\left[\pi - \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)\right] = -\frac{2\pi}{3}$

$z = 2e^{i\left(-\frac{2\pi}{3}\right)}$

$$\frac{(iz)^n}{z^2} = \frac{e^{i\left(\frac{n\pi}{2}\right)} 2^n e^{i\left(-\frac{2n\pi}{3}\right)}}{2^2 e^{i\left(-\frac{4\pi}{3}\right)}}$$

$$= 2^{n-2} e^{i\left(\frac{n\pi}{2} - \frac{2n\pi}{3} + \frac{4\pi}{3}\right)}$$

$$= 2^{n-2} e^{i\left(\frac{(8-n)\pi}{6}\right)}$$

$\frac{(iz)^n}{z^2}$ is purely imaginary: $\cos\left(\frac{(8-n)\pi}{6}\right) = 0$

$$\frac{(8-n)\pi}{6} = (2k+1)\frac{\pi}{2},\ k \in \mathbb{Z}$$

$$n = 5 - 6k,\ k \in \mathbb{Z}$$

Note: You may also have alternative form:

$$\frac{(8-n)\pi}{6} = (2k-1)\frac{\pi}{2},\ k \in \mathbb{Z}$$

$$n = 11 - 6k,\ k \in \mathbb{Z}$$

$\therefore$ smallest positive integer $n = 5$.

Alternative Method:

$$n\arg(iz) - 2\arg(z) = n\arg(i) + n\arg(z) - 2\arg(z)$$

$$= \frac{n\pi}{2} - \frac{2n\pi}{3} + \frac{4\pi}{3}$$

$$= \frac{(8-n)\pi}{6}$$

(ii)

$|wz| = 4$

$2|w| = 4$

$|w| = 2$

$\arg\left(\frac{w^*}{z^2}\right) = -\frac{5\pi}{6}$

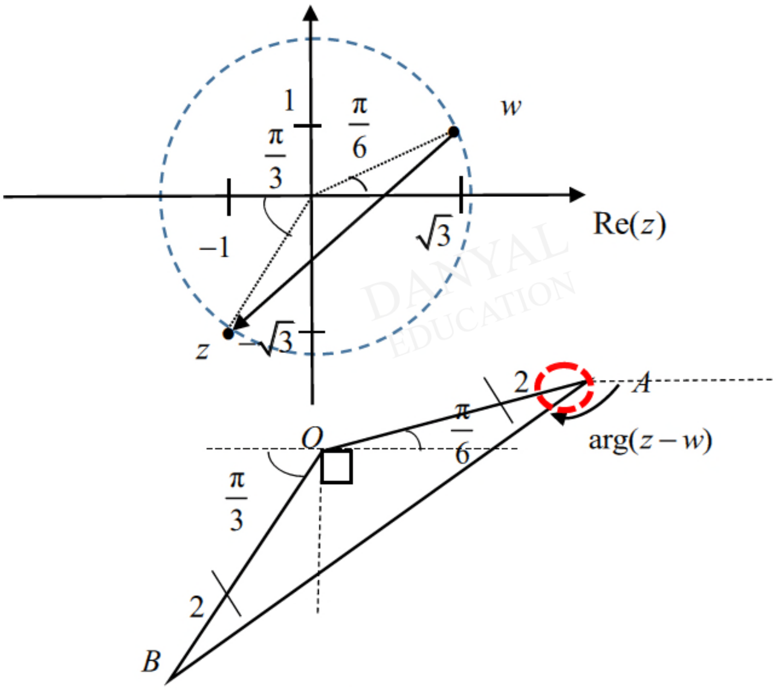
$-\arg(w) - 2\arg(z) = -\frac{5\pi}{6}$

$\arg(w) = \frac{5\pi}{6} - 2\left(-\frac{2\pi}{3}\right)$

$= \frac{13\pi}{6}$

Since $-\pi < \arg(w) \leq \pi$, $\arg(w) = \frac{\pi}{6}$ (exact).

(iii)



$\angle OAB = \frac{1}{2}\left\{\pi - \left[\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + \frac{\pi}{2} + \frac{\pi}{6}\right]\right\} = \frac{\pi}{12}$

Hence Method: $\arg(z-w) = -\left[\pi - \frac{\pi}{6} - \frac{\pi}{12}\right]$

$= -\left[\frac{5\pi}{6} - \left(\frac{1}{2}\left\{\pi - \frac{5\pi}{6}\right\}\right)\right]$

$= -\frac{3\pi}{4}$ (exact)

Otherwise Method:

$z - w = \left(-1-\sqrt{3}\right) + \left(-1-\sqrt{3}\right)\mathrm{i}$ $\qquad \arg(z-w) = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$