A Level H2 Math

Binomial Distribution Test 5

Q1

In a large company, a small sample of n employees is obtained to find out their mode of transport to work. The number of employees who ride the train to work is denoted by R. Assume that R has the distribution $\mathrm{B}(n, p)$.

(i) Given that $n=10$, find the value of p if the probability that 6 employees ride the train to work is twice the probability that 4 employees ride the train to work. [3]

(ii) Given that $p=0.25$, find the largest value of n such that the probability that fewer than 2 employees who ride the train to work is more than 0.15. [3]

(iii) Given that $n=11$ and $p=0.7$, find the probability that at least 5 employees ride the train to work if at least 3 employees do not ride the train to work. [4]

Q2

Coloured lego pieces are packed into boxes of 20 pieces by a particular manufacturer. Each box is made up of randomly chosen coloured lego pieces. The manufacturer produces a large number of lego pieces every day. On average, 15% of lego pieces are red. Explain why binomial distribution is appropriate for modelling the number of red lego pieces in a box. [2]

(i) Find the probability that a randomly chosen box of lego pieces contains at least 4 red lego pieces. [2]

(ii) A customer buys 50 randomly chosen boxes containing lego pieces. Find the probability that no more than 19 of these boxes contain at least 4 red lego pieces. [2]

It is given instead that the proportion of lego pieces that are red is now p. The probability that there is at least one red lego piece but fewer than four red lego pieces in a box, is 0.22198, correct to 5 significant figures. Write down an equation involving p and hence find the value of p, given that $p > 0.2$. [4]

Q3

A factory manufactures large number of pen refills. From past records, 3% of the refills are defective. A stationery store manager wishes to purchase pen refills from the factory. To decide whether to accept or reject a batch of refills, the manager designs a sampling process. He takes a random sample of 25 refills. The batch is accepted if there is no defective refill and rejected if there are more than 2 defective refills. Otherwise, a second random sample of 25 refills is taken. The batch is then accepted if the total number of defective refills in the two samples is fewer than 4 and rejected otherwise.

(i) Find the probability of accepting a batch. [4]

(ii) If a batch is accepted, find the probability that there are 2 defective refills found in the sampling process. [3]

The stationery store manager purchases 50 boxes of 25 refills each.

(iii) Find the probability that the mean number of defective refills in a box is less than 1. [2]

Answers

Binomial Distribution Test 5

Q1

(i)

$$P(R=6)=2P(R=4)$$

$$^{10}C_6 p^6 (1-p)^4 = 2\,^{10}C_4 p^4 (1-p)^6$$

$$p^2 = 2(1-p)^2$$

$$p^2 - 4p + 2 = 0$$

$$p = 0.586$$

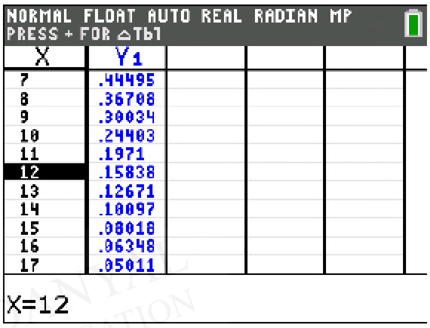
(ii)

$$R \sim B(n, 0.25)$$

$$P(R<2) > 0.15$$

$$P(R \le 1) > 0.15$$

$$n = 12$$



(iii)

$$R \sim B(11, 0.7)$$

$$P(R \ge 5 \mid R \le 8) = \frac{P(R \ge 5 \text{ and } R \le 8)}{P(R \le 8)}$$

$$= \frac{P(5 \le R \le 8)}{P(R \le 8)}$$

$$= \frac{P(R \le 8) - P(R \le 4)}{P(R \le 8)}$$

$$= 0.969$$

2 major errors seen in the students' scripts.
1. Failure to recognise that this is a conditional probability.
2. Failure to count the number of cases for the numerator.

Q2

A binomial distribution is appropriate as there is a large number of lego pieces with constant probability 0.15 of them being red suggests independence in selection. Moreover, there are only two possible outcomes (red or non red).

(i) Let X be the number of lego pieces, out of 20, that are red.

$X \sim \mathrm{B}(20, 0.15)$

$$\begin{aligned}\mathrm{P}(X \geq 4) &= 1 - \mathrm{P}(X \leq 3) \\ &= 0.35227 \\ &= 0.352 \quad (3\text{s.f.})\end{aligned}$$

(ii) Let Y be the number of boxes of lego pieces, out of 50, that contain at least 4 red lego pieces.

$Y \sim \mathrm{B}(50, 0.35227)$

$$\begin{aligned}\mathrm{P}(Y \leq 19) &= 0.71498 \\ &= 0.715 \quad (3\text{ s.f.})\end{aligned}$$

(iii) Let A be the number of lego pieces, out of 20, that are red.

$A \sim \mathrm{B}(20, p)$

$\mathrm{P}(1 \leq A < 4) = 0.22198$

$\mathrm{P}(A=1) + \mathrm{P}(A=2) + \mathrm{P}(A=3) = 0.22198$

$$\binom{20}{1} p(1-p)^{19} + \binom{20}{2} p^2(1-p)^{18} + \binom{20}{3} p^3(1-p)^{17} = 0.22198$$

$$20p(1-p)^{19} + 190p^2(1-p)^{18} + 1140p^3(1-p)^{17} = 0.22198$$

Since $0.2 < p < 1$, $p = 0.250$ (3 s.f)

Q3

Let X be the number of defective refills in the sample of 25 refills drawn from a batch which contains 3% defective refills. Then, $X \sim \mathrm{B}(25, 0.03)$ (i) P(accepting a batch) $= \mathrm{P}(X=0)+\mathrm{P}(X=1)\mathrm{P}(X\le 2)+\mathrm{P}(X=2)\mathrm{P}(X\le 1)$ $= 0.4669747053 + 0.3473570958 + 0.1109593034$ ≈ 0.9252911 $= 0.925$ (correct to 3 s.f.)	The cases in which the batch can be accepted should be thought through carefully.
(ii) Required probability = P(2 defective refills | batch is accepted) $= \dfrac{\mathrm{P}(X_1=1)\mathrm{P}(X_2=1)+\mathrm{P}(X_1=2)\mathrm{P}(X_2=0)}{0.9252911}$ $=0.209$ (correct to 3 s.f.)	The question is asking for the conditional probability of having 2 defective refills given that the batch is accepted.

(iii) $X \sim \mathrm{B}(25, 0.03)$ Since *sample size* = 50 is large, by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(25(0.03), \dfrac{25(0.03)(0.97)}{50}\right)$ approximately $\bar{X} \sim \mathrm{N}(0.75, 0.1455)$ Required probability $= P(\bar{X} < 1) = 0.981$ (correct to 3 s.f.) **Alternative solution** $X_1 + \ldots + X_{50} \sim \mathrm{B}(50 \times 25, 0.03)$ $X_1 + \ldots + X_{50} \sim \mathrm{B}(1250, 0.03)$ $\mathrm{P}(X_1 + \ldots + X_{50} < 50)$ $= \mathrm{P}(X_1 + \ldots + X_{50} \le 49)$ $= 0.973$	In general, when the question asks for a mean number of X when X is a discrete random variable, students should consider applying Central Limit Theorem. This is especially so if the question has keywords such as "approximate/estimate the probability".

Marker's comments

Part (i):

1. Quite a large number of students did not understand the first line and hence did not realise that the number of defective refills follow a Binomial Distribution. This leads to an attempt to list out all the cases manually. While computing the individual cases, most students using this approach did not consider the order of appearance of the "defective" refills (as per Binomial formula).

2. For students who considered the Binomial Distribution, many did not understand the selection process if a second batch is required. Many took the question at face value, i.e. $\mathrm{P}(1 \le X_1 \le 2) \cdot \mathrm{P}(X_1 + X_2 < 4)$. Students need to realise that the number of defects in the first sample affects the allowable number of defects in the second sample.

Part (ii):

3. Apart from not realising that the question is asking for the conditional probability, many students were unable to identify the cases of having P(2 defective refills $\cap$ batch is accepted). They either forgot that we can have $\mathrm{P}\left(X_1 = 2\right)\mathrm{P}\left(X_2 = 0\right)$, or thought that $\mathrm{P}\left(X_1 = 0\right)\mathrm{P}\left(X_2 = 2\right)$ was possible. The latter is not possible because if $\mathrm{P}(X_1 = 0)$, there the sample would have been accepted immediately and a second sample would not be taken.

Part (iii):

4. The computation of the parameters for $\overline{X}$ was poorly done. There was a lot of confusion about what n is. In this case, $X \sim$ B (25, 0.03) and we have 50 samples. Hence $X_1 + \ldots + X_{50} \sim N(50 \times 25 \times 0.03, 50 \times 25 \times 0.03 \times 0.97)$ approx. by CLT, and hence $\overline{X} = \dfrac{X_1 + \ldots + X_{50}}{50} \sim N(25 \times 0.03, \dfrac{25 \times 0.03 \times 0.97}{50})$.