A Level H2 Math

Binomial Distribution Test 3

Q1

In a large shipment of glass stones used for the *Go* board game, a proportion p of the glass stones is chipped. The glass stones are sold in boxes of 361 pieces each. Let X denote the number of chipped glass stones in a box.

(i) Based on this context, state two assumptions in order for X to be well modelled by a binomial distribution. [2]

In the rest of the question, assume that X follows a binomial distribution.

(ii) It is known that the probability of a box containing at most 2 chipped glass stones is 0.90409. Find p. [2]

(iii) A box is deemed to be of inferior quality if it contains more than 2 chipped glass stones. Find the probability that, in a batch of 20 boxes of glass stones, there are more than 5 boxes of inferior quality in the batch. [3]

(iv) Each week, a distributor purchases 50 batches of glass stones, each batch consisting of 20 boxes of glass stones. A batch will be rejected if it contains more than 5 boxes of inferior quality. The distributor will receive a compensation of \$10 for each rejected batch in the first 20 weeks of a year, and a compensation of \$20 for each rejected batch in the remaining weeks of the year. Assuming that there are 52 weeks in a year, find the probability that the total compensation in a year is more than \$250. [5]

Q2

A jar contains 5 identical balls numbered 1 to 5. A fixed number, n, of balls are selected and the number of balls with an even score is denoted by X.

(i) Explain how the balls should be selected in order for X to be well modelled by a binomial distribution. [2]

Assume now that X has the distribution $\mathrm{B}\left(n, \frac{2}{5}\right)$.

(ii) Given that $n = 10$, find $\mathrm{P}(X \geq 4)$. [2]

(iii) Given that the mean of X is 4.8, find n. [2]

(iv) Given that $\mathrm{P}(X = 0 \text{ or } 1) < 0.01$, write down an inequality for n and find the least value of n. [3]

Shawn and Arvind take turns to draw one ball from the jar at random. The first person who draws a ball with an even score wins the game. Shawn draws first.

(v) Show that the probability that Shawn wins the game is $\frac{3}{5}$ if the selection of balls is done without replacement. [2]

(vi) Find the probability that Shawn wins the game if the selection of balls is done with replacement. [2]

Q3

A biscuit manufacturer produces both cream and chocolate biscuits. Biscuits are chosen randomly and packed into boxes of 10. The number of cream biscuits in a box is denoted by X.

(a) On average, the proportion of cream biscuits is p. Given that $P(X = 1 \text{ or } 2) = 0.15$, write down an equation for the value of p. Hence find the value(s) of p numerically. [3]

(b) It is given instead that the biscuit manufacturer produces 3 times as many cream biscuits as chocolate biscuits.

(i) Find the most likely value of X. [2]

(ii) A random sample of 18 boxes is taken. Find the probability that at least 3 but fewer than 7 boxes have equal numbers of cream and chocolate biscuits. [3]

A box of biscuits is sold at \$10. The manufacturer gives a discount of \$2 per box to its premium customers. The mean and variance of the number of boxes sold per day to each type of customers (assuming independence) are as follows:

	Mean	Variance
Number of boxes sold at usual price	180	64
Number of boxes sold at discounted price	840	169

Find the approximate probability that the total amount collected per month from the sales of biscuits is not less than \$255,000, assuming that there are 30 days in a month. [4]

Answers

Binomial Distribution Test 3

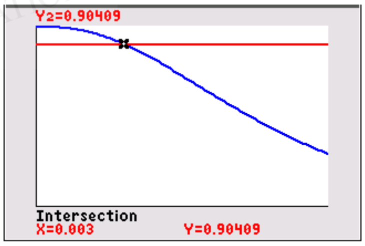
Q1

(i)
Assumptions
- The probability of a randomly chosen glass stone being chipped is constant.
- Whether a glass stone is chipped or not is independent of that of any other glass stones.

(ii)
$X \sim \mathrm{B}(361, p)$

$\mathrm{P}(X \le 2) = 0.90409$

Using GC,
$p = 0.00300$



(iii)
$P(X > 2) = 1 - P(X \le 2) = 1 - 0.90409 = 0.09591$

Let Y be number of boxes with more than 2 chipped glass stones, out of 20 boxes.

$$Y \sim \mathrm{B}(20, 0.09591)$$

$$\begin{aligned}\mathrm{P}(Y > 5) &= 1 - \mathrm{P}(Y \le 5)\\ &= 1 - 0.9907736392\\ &= 0.0092263608\\ &\approx 0.00923\end{aligned}$$

(iv)
Let A be the number of rejected batches, out of 50 batches.

$$A \sim \mathrm{B}(50, 0.0092263608)$$

$\mathrm{E}(A) = 50(0.0092264) = 0.46132$

$\mathrm{Var}(A) = 50(0.0092264)(1 - 0.0092264) = 0.45706$

Let $M_1 = A_1 + \ldots + A_{20}$

Since $n = 20$ is sufficiently large, by CLT,

$M_1 \sim \mathrm{N}(20 \times 0.46132,\ 20 \times 0.45706)$

$= \mathrm{N}(9.2264,\ 9.1412)$ approximately

Let $M_2 = A_{21} + \ldots + A_{52}$

Since $n = 32$ is sufficiently large, by CLT,

$M_2 \sim \mathrm{N}(32 \times 0.46132,\ 32 \times 0.45706)$

$= \mathrm{N}(14.76224,\ 14.62592)$ approximately

Let $T = 10M_1 + 20M_2$

Hence $T \sim \mathrm{N}\left(10(9.2264) + 20(14.76224),\ 10^2(9.1412) + 20^2(14.62592)\right)$

$= \mathrm{N}(387.5088, 6764.488)$ approximately

$$\therefore\ \mathrm{P}(T > 250) = 0.952729 \approx 0.953$$

Q2

(i)

(1) Selection of balls is done with replacement.

(2) The balls are thoroughly mixed before each selection.

(ii) Given $X \sim \mathrm{B}\left(10, \frac{2}{5}\right)$

$$\mathrm{P}(X \geq 4) = 1 - \mathrm{P}(X \leq 3) = \underline{\underline{0.618}} \text{ (3 sf)}$$

(iii) Given

$$\mathrm{E}(X) = 4.8$$

$$\Rightarrow \frac{2}{5}n = 4.8$$

$$n = \underline{\underline{12}}$$

(iv) Given $X \sim \mathrm{B}\left(n, \frac{2}{5}\right)$

$$\mathrm{P}(X = 0 \text{ or } 1) < 0.01$$

$$\Rightarrow \mathrm{P}(X=0) + \mathrm{P}(X=1) < 0.01$$

$$\Rightarrow \left(\frac{3}{5}\right)^n + n\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^{n-1} < 0.01$$

From GC, least $n = \underline{\underline{14}}$

(v) Without replacement,

P(Shawn wins the game)

$$= \frac{2}{5} + \frac{3}{5}\left(\frac{2}{4}\right)\left(\frac{2}{3}\right)$$

$$= \frac{3}{5} \quad \text{(Shown)}$$

(vi) With replacement,

P(Shawn wins the game)

$$= \frac{2}{5} + \frac{3}{5}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right) + \frac{3}{5}\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right) + \ldots$$

$$= \frac{2}{5} + \frac{2}{5}\left(\frac{3}{5}\right)^2 + \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^4 + \ldots$$

$$= \frac{\frac{2}{5}}{1 - \left(\frac{3}{5}\right)^2}$$

$$= \underline{\underline{\frac{5}{8}}} \text{ or } \underline{\underline{0.625}}$$

Q3

8 (a)

Let X be the number of cream biscuits per box. $X \sim B(10, p)$

$P(X = 1 \text{ or } 2) = 0.15$

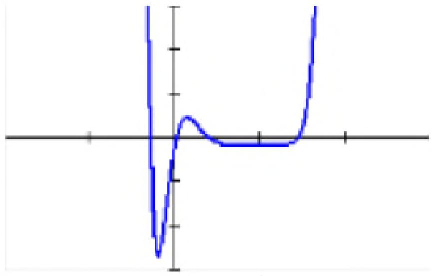
$P(X = 1) + P(X = 2) = 0.15$

$^{10}C_1 p^1(1-p)^9 + {}^{10}C_2 p^2(1-p)^8 = 0.15$

$10p(1-p)^9 + 45p^2(1-p)^8 = 0.15$

$5p(1-p)^8\left[2(1-p)+9p\right] = 0.15$

$5p(1-p)^8\left(2+7p\right) = 0.15$



Draw $y = 5p(1-p)^8(2+7p) - 0.15$

From G.C.,
$p = 0.0162$ or $p = 0.408$
(other values are 1.45 or -0.288 need to be rejected)

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8 (b) (i)

$X \sim B(10, \frac{3}{4})$. Let $Y_1 = P(X = x)$.

X	Y1
0	9.5E-7
1	2.9E-5
2	3.9E-4
3	.00309
4	.01622
5	.0584
6	.146
7	.25028
8	.28157
9	.18771
10	.05631

From G.C.,
since $P(X = 8)$ is the highest,
The most likely no. of cream biscuits = 8

(ii)

Let Y denote the random variable: Number of boxes with X = 5.

$Y \sim B(18, p)$ where $p = P(X=5) = 0.058399$

$P(3 \le Y < 7) = P(Y \le 6) - P(Y \le 2)$

$= 0.0843$ (3 s.f.)

(iii)

Let U = no. of boxes sold at Usual price
Let D = no. of boxes sold at Discounted price
Let W: Total income per day.

$W = 10U + 8D$

$E(W) = 10E(U) + 8E(D) = 180 \times \$10 + 840 \times \$8 = \$8520$

$Var(W) = 10^2 Var(U) + 8^2 Var(D) = 64 \times 10^2 + 169 \times 8^2 = 17216$

Let $T = W_1 + W_2 + \ldots + W_{30}$

Since n = 30 is large, by Central Limit Theorem,

$T \sim N(30 \times 8520, 30 \times 17216) = N(255600, \sqrt{516480}^2)$ approximately

$P(T \ge \$255000) = 0.798$