A Level H2 Math

Binomial Distribution Test 2

Q1

(a) The random variable X follows a binomial distribution $\mathrm{B}(10, p)$.

(i) Given that X has two modes, $X = 4$ and $X = 5$, find the exact value of p. [2]

(ii) Given instead that $\mathrm{P}(X \le 9) = \dfrac{1023}{1024}$, find the exact value of p. [2]

(b) The random variable Y follows a binomial distribution $\mathrm{B}(500, 0.5)$.
A sample of 30 independent values of Y is recorded.

(i) Find the probability that all the values recorded are less than or equal to 256. [2]

(ii) The mean of the 30 values is calculated. Estimate the probability that this sample mean is less than or equal to 256, stating clearly the approximation used. [3]

(iii) Explain why the probability found in part **(ii)** is larger than that found in part **(i)**. [1]

Q2

Factory A manufactures a large batch of light bulbs. It is known that on average, 1 out of 200 light bulbs manufactured by Factory A, is defective. A random sample of 180 light bulbs is inspected. The batch is accepted if the sample contains less than r defective light bulbs.

(i) Explain why the context above may not be well-modelled by a binomial distribution. [1]

Assume now that the context above is well-modelled by a binomial distribution.

(ii) Determine the value of r such that the probability of accepting the batch is 0.998. [1]

In Factory B, a random sample of 30 light bulbs is taken from a large batch. If the sample contains no defective light bulbs, the batch is accepted. The batch is rejected if the sample contains more than two defective light bulbs. If the sample contains one or two defective light bulbs, a second random sample of 30 light bulbs is chosen and the batch is accepted only if this second sample contains no defectives. It is known that Factory B produces $(100p)\%$ defective light bulbs.

(iii) Find the probability that the batch is accepted. Leave your answer in terms of p. [3]

Forty random samples of 30 light bulbs are taken from each of the two factories A and B.

(iv) Given that $p = 0.007$ and there is exactly one defective bulb, find the probability that it is from Factory B. [4]

Q3

A sample of 5 people is chosen from a village of large population.

(i) The number of people in the sample who are underweight is denoted by X. State, in context, the assumption required for X to be well modelled by a binomial distribution. [1]

(ii) On average, the proportion of people in the village who are underweight is p. It is known that the mode of X is 2. Use this information to show that $\frac{1}{3} < p < \frac{1}{2}$. [3]

1000 samples of 5 people are chosen at random from the village and the results are shown in the table below.

x	0	1	2	3	4	5
Number of groups	93	252	349	220	75	11

(iii) Using the above results, find $\bar{x}$. Hence estimate the value of p. [2]

You may now use your estimate in part **(iii)** as the value of p.

(iv) Two random samples of 5 people are chosen. Find the probability that the first sample has at least 4 people who are underweight and has more people who are underweight than the second sample. [3]

Answers

Binomial Distribution Test 2

Q1

(a)(i)	$\mathrm{P}(X=4)=\mathrm{P}(X=5)$ $\frac{10!}{4!6!}p^4(1-p)^6=\frac{10!}{5!5!}p^5(1-p)^5$ $5(1-p)=6p$ $p=\frac{5}{11}$	Most students could identify that the probabilities for the outcomes of 4 and 5 should be equal and wrote the expressions according to the formula. Some failed to solve the equation due to inadequate skills in algebra.
(ii)	$\mathrm{P}(X\le 9)=\frac{1023}{1024}$ $\mathrm{P}(X=10)=\frac{1}{1024}$ $p^{10}=\left(\frac{1}{2}\right)^{10}$ $p=\frac{1}{2}$	Many students did not realise that the complementary case is simply 10. Once this hurdle was overcome most were able to find the final answer.
(b)(i)	$\mathrm{P}(Y\le 256)=0.719485301$ P(all 100 values are less than or equal to 256) $=0.719485301^{30}$ $=0.0000514$	Many students immediately dived into the irrelevant routine of using CLT to find the sampling distribution once they saw the conditions given, without analyzing the question carefully. Majority of the students left the first probability as the answer. Their understanding of the term "sample" may be in question.

(ii)	$\mathrm{E}(Y) = 500(0.5) = 250$, and $\mathrm{Var}(Y) = 500(0.5)(0.5) = 125$ Since the sample size is sufficiently large, $\bar{Y} \sim \mathrm{N}\left(250, \frac{125}{30}\right)$ approximately by CLT $\mathrm{P}\left(\bar{Y} \le 256\right) = 0.998$	Most were able to follow the routine to write down the expectation and variance of Y. However, half of them did not show clear understanding of sampling distribution and central limit theorem in their subsequent presentation of the solution. The most common mistake is that quoting CLT to write down $Y \sim N(250,125)$, which is WRONG! It is the mean of samples of large size may be considered as normally distributed approximately, not the individual observation. Other common mistakes include forgetting to divide the variance by sample size, or using a wrong notation for the random variable of sample mean.
(iii)	The probability in part **(ii)** included cases where some of the values can be larger than 256, but the final average is still at most 256.	Many students were able to give the correct reason though the phrasing can still be improved. For example many casually wrote “probability in (i) is a subset of probability in (ii)”, which showed understanding but failed to make sense mathematically when it is the collection of “events/outcomes” in one being subset of the other.

Q2

(i)

The event of a bulb being defective may not be independent of another bulb being defective.

(ii)

Let X be the random variable for the number of defective light bulbs produced by Factory A.

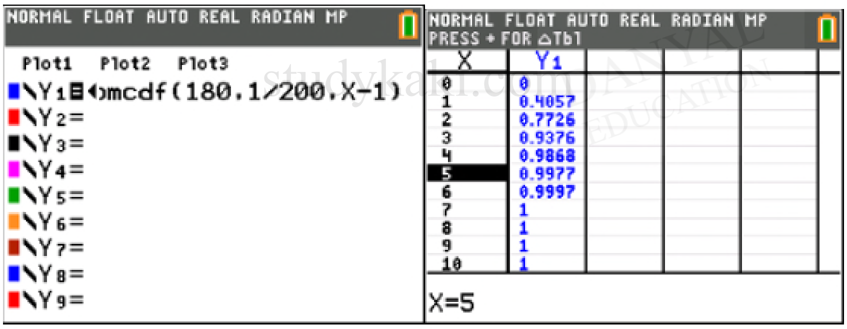
$$X \sim \mathrm{B}\left(180, \frac{1}{200}\right)$$

Given

$$\mathrm{P}(X < r) = 0.998$$

$$\Rightarrow \mathrm{P}(X \leq r - 1) = 0.998$$

By GC,



$\therefore r = 5$

(iii)

Let Y be the random variable for the number of defective light bulbs produced by Factory B.

$Y \sim \mathrm{B}(30, p)$

P(the batch is accepted)

$= \mathrm{P}(Y_1 = 0) + \mathrm{P}(Y_1 = 1 \text{ or } 2)\mathrm{P}(Y_2 = 0)$

$$= \binom{30}{0} p^0 (1-p)^{30}$$

$$+ \left[\binom{30}{1} p(1-p)^{29} + \binom{30}{2} p^2 (1-p)^{28} \right] \binom{30}{0} p^0 (1-p)^{30}$$

$$= (1-p)^{30} + \left[30p(1-p)^{29} + 435p^2(1-p)^{28} \right](1-p)^{30}$$

$$= (1-p)^{30} + 30p(1-p)^{59} + 435p^2(1-p)^{58}$$

(iv)

Let U be the random variable for the number of defective light bulbs produced by Factory A.

Let V be the random variable for the number of defective light bulbs produced by Factory B.

$U \sim \mathrm{B}\left(1200, \frac{1}{200}\right)$

$V \sim \mathrm{B}(1200, 0.007)$

P(1 bulb from B is defective| there is exactly one defective bulb)

$$=\frac{\mathrm{P}(U=0,V=1)}{\mathrm{P}(U=0,V=1)+\mathrm{P}(U=1,V=0)}$$

$$\approx 0.5838223$$

$$=0.584\text{ (3 s.f.)}$$

Reference for $\dfrac{\mathrm{P}(U=0,V=1)}{\mathrm{P}(U=0,V=1)+\mathrm{P}(U=1,V=0)}$:

$$\frac{\left[\binom{1200}{1}0.007^{1}(1-0.007)^{1199}\times\binom{1200}{0}\left(\frac{1}{200}\right)^{0}\left(1-\frac{1}{200}\right)^{1200}\right]}{\left[\begin{array}{l}\binom{1200}{1}0.007^{1}(1-0.007)^{1199}\times\binom{1200}{0}\left(\frac{1}{200}\right)^{0}\left(1-\frac{1}{200}\right)^{1200}\\+\binom{1200}{0}0.007^{0}(1-0.007)^{1200}\times\binom{1200}{1}\left(\frac{1}{200}\right)^{1}\left(1-\frac{1}{200}\right)^{1199}\end{array}\right]}$$

Q3

(i) Assume that the:

- weights of the 5 people chosen are independent of each other
- sample is chosen randomly.

(ii) $P(X=1) < P(X=2)$
and $P(X=2) > P(X=3)$

$^5C_1 p(1-p)^4 < {}^5C_2 p^2(1-p)^3$
and $^5C_2 p^2(1-p)^3 > {}^5C_3 p^3(1-p)^2$

Since $(1-p) > 0$ and $p > 0$,

$1-p < 2p$ and $1-p > p$

$p > \frac{1}{3}$ and $p < \frac{1}{2}$

$\therefore \frac{1}{3} < p < \frac{1}{2}$ (shown)

(iii) $\bar{x} = 1.965$ (from GC)

Since $n = 5$, $np \approx 1.965 \Rightarrow p \approx 0.393$

(iv) $X \sim B(5, 0.393)$

$P((X_1 \geq 4) \cap (X_1 > X_2))$

$= P(X_1 = 4)P(X_2 \leq 3) + P(X_1 = 5)P(X_2 \leq 4)$

$= 0.0724(0.91823) + (0.00937)(0.99063)$

$= 0.0758$ (3 sf)