A Level H2 Math

Binomial Distribution Test 1

Q1

(i) A procedure for accepting or rejecting a large batch of manufactured articles is such that an inspector first selects and examines a random sample of 10 articles from the batch. If the sample contains at least 2 defective articles, the batch is rejected.
It is known that the proportion of articles that are defective is 0.065. Show that the probability that a batch of articles is accepted is 0.866, correct to three significant figures. [1]
To confirm the decision, another inspector follows the same procedure with another random sample of 10 articles from the batch. If the conclusion of both inspectors are the same, the batch will be accepted or rejected as the case may be. Otherwise, one of the inspectors will select a further random sample of 10 from the same batch to examine. The batch is then rejected if there are at least 2 defective articles. Otherwise, it is accepted. Find

(a) the probability that a batch is eventually accepted, [3]

(b) the expected number of articles examined per batch. [4]

(ii) In order to cut labour cost, an alternative procedure is introduced. A random sample of 10 articles is taken from the batch and if the sample contains not more than 1 defective article then the batch is accepted. If the sample contains more than 2 defective articles, the batch is rejected. If the sample contains exactly 2 defective articles, a second sample of 10 articles is taken and if this contains no defective article then the batch is accepted. Otherwise, the batch is rejected. Given that the proportion of defective articles in the batch is p, show that the probability that the batch is accepted is A where

$$A = (1+9p)(1-p)^9 + 45p^2(1-p)^{18}.$$ [2]

If the probability that, of 100 batches inspected, more than 80 of them will be accepted is 0.98, find the value of p. [3]

Q2

A geologist splits rocks to look for fossils. On average 7% of the rocks selected from a particular area contain fossils.

The geologist selects a random sample of 20 rocks from this area.

(i) Find the probability that at least three of the rocks contain fossils. [2]

A random sample of n rocks is selected from this area.

(ii) The geologist wants to have a probability of 0.8 or greater of finding fossils in at least three of these rocks. Find the least possible value of n. [3]

In early 2017, geologists found the fossils of *zilantophis schuberti*, a new discovered species of winged serpent. On average, the proportion of rocks that contain fossils of *zilantophis schuberti* in this area is p. It is known that the modal number of fossils of *zilantophis schuberti* in a random sample of 10 rocks is 3.

(iii) Use this information to find exactly the range of values that p can take. [4]

Q3

A car park next to a small commercial building has a total of 12 parking lots. Land surveillance officers have been observing the usage of parking lots per day to determine if the land has been efficiently utilised. Each parking lot can be occupied by at most one vehicle per day.

(i) Denoting the number of occupied parking lots per day by X, state in context, two assumptions needed for X to be well modelled by a binomial distribution. [2]

(ii) It is further observed that for 80% of the days in the survey period, there are at least 4 occupied lots in the car park for each day. Find the probability that a parking lot is being occupied in a day. [2]

(iii) Given that at least one of the parking lots is occupied in a particular day, show that the probability that at least 2 but less than 4 lots are occupied in the particular day is given by

$$f(p) = \frac{22p^2(1-p)^9(3+7p)}{1-(1-p)^{12}}$$

where p is the probability of a parking lot being occupied in a day. What can you say about this probability if p is approximately 0.185? [5]

Answers

Binomial Distribution Test 1

Q1

(a)	Let X be the random variable 'number of defective articles in sample of 10'. $X \sim \mathrm{B}(10, 0.065)$ $\mathrm{P}(\text{accepting a batch}) = \mathrm{P}(X \le 1) = 0.86563 = 0.866$	Although most people are able to do this part, there are quite a number of students who doesn't know how to do this basic question. Or some calculated this manually instead of using Binomial distribution.
(i)	P(batch eventually accepted) $= (0.86563)^2 + 2(0.86563)(1-0.86563)(0.86563)$ $= 0.95069$ $= 0.951$	Most students who got this wrong did not multiply by 2 for the second case. Some did not understand the question and interpret it as a geometric series question.
(ii)	Let N be the number of articles examined per batch. $N = \begin{cases} 20 & \text{if both findings agree} \\ 30 & \text{otherwise} \end{cases}$ $\mathrm{P}(N=20) = (0.86563)^2 + (1-0.86563)^2 = 0.76737$ $\mathrm{P}(N=30) = 1 - 0.76737 = 0.23263$ $\therefore\ \mathrm{E}(N) = 20(0.76737) + 30(0.23263) = 22.3$	About 30% have no clue how to do this part. 40% of those who attempted missed out some cases, such as RR or did not multiply by 2 to account for AR and RA.

(b)	Let Y be the random variable 'number of defective articles in a sample of 10'. $Y \sim \mathrm{B}(10, p)$ $A = \mathrm{P}(Y \le 1) + \mathrm{P}(Y = 2) \cdot \mathrm{P}(Y = 0)$ $= {}^{10}C_0 p^0 (1-p)^{10} + {}^{10}C_1 p^1 (1-p)^9 + {}^{10}C_2 p^2 (1-p)^8 \cdot {}^{10}C_0 p^0 (1-p)^{10}$ $= (1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^{18}$ $= (1+9p)(1-p)^9 + 45p^2(1-p)^{18}$ (shown)	Except for some who did not interpret the question properly, this part is quite well done for those who attempted it. Except for those who did not use the formula and thus left out ${}^{10}C_1$ or ${}^{10}C_2$.
(b)	Let W be the random variable 'number of acceptable batches, out of 100 inspected'. $W \sim \mathrm{B}(100, A)$ $\mathrm{P}(W > 80) = 0.98 \Rightarrow \mathrm{P}(W \le 80) = 0.02$ By GC, $A = 0.876235$ $\therefore A = (1+9p)(1-p)^9 + 45p^2(1-p)^{18} = 0.87624$ By GC, $p = 0.08$	There are a good number students who have problem dealing with complement. $\mathrm{P}(W > 80) = 0.98 \Rightarrow 1 - \mathrm{P}(W \le 79) = 0.98$ A large number of students applied (CLT) erroneously or normal approximation to this qn, and took invNorm. Students should also be advised not to use table to solve for A as A is not an integer value.

Q2

(i) Let X be the number of rocks containing fossils out of 20 rocks.

$X \sim \text{B}(20,\ 0.07)$

$\text{P}(X \geq 3) = 1 - \text{P}(X \leq 2)$

$= 0.161$ (3 s.f.)

(ii) Let Y be the number of rocks containing fossils out of 20 rocks.

$Y \sim \text{B}(n,\ 0.07)$

$\text{P}(Y \geq 3) \geq 0.8$

Method 1a: Using GC Table

n	$\text{P}(Y \geq 3)$	
59	0.79085	< 0.8
60	0.80023	> 0.8
61	0.80925	> 0.8

Hence, least $n = 60$.

Method 1b: Using GC Table

$\text{P}(Y \leq 2) \leq 0.2$

n	$\text{P}(Y \leq 2)$	
59	0.20915	> 0.2
60	0.19977	< 0.2
61	0.19075	< 0.2

Hence, least $n = 60$.

Method 2: Using the binomial distribution function

$\text{P}(Y \leq 2) \leq 0.2$

$\text{P}(Y = 0) + \text{P}(Y = 1) + \text{P}(Y = 2) \leq 0.2$

$$0.93^n + n(0.07)(0.93)^{n-1} + \frac{n(n-1)}{2}(0.07^2)(0.93)^{n-2} \leq 0.2$$

Using GC to sketch the graph:

Hence, least $n = 60$.

(iii) Let W be the number of fossils of *zilantophis schuberti* in a random sample of 10 rocks.
$W \sim \mathrm{B}(10,\ p)$

$$\mathrm{P}(W=3) > \mathrm{P}(W=2)$$
$$\frac{10!}{3!7!}p^3(1-p)^7 > \frac{10!}{2!8!}p^2(1-p)^8$$
$$120p^3(1-p)^7 > 45p^2(1-p)^8$$
$$8p > 3(1-p) \qquad (\text{Since } 0<p<1)$$
$$\frac{8}{3}p > 1-p$$
$$p > \frac{3}{11}$$

$$\mathrm{P}(W=3) > \mathrm{P}(W=4)$$
$$\frac{10!}{3!7!}p^3(1-p)^7 > \frac{10!}{4!6!}p^4(1-p)^6$$
$$120p^3(1-p)^7 > 210p^4(1-p)^6$$
$$4(1-p) > 7p \qquad (\text{Since } 0<p<1)$$
$$1-p > \frac{7}{4}p$$
$$p < \frac{4}{11}$$

$$\therefore \frac{3}{11} < p < \frac{4}{11}$$

Q3

(i)

The 2 assumptions needed for X to be well modelled by a binomial distribution are as follow:

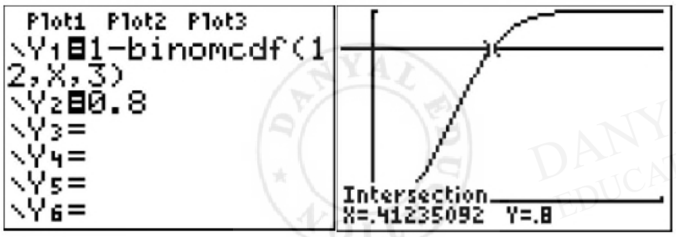
1. The occupancy of any particular parking lot in the car park is *independent* of that of another lot.
2. The probability of a parking lot being occupied in a day is *constant* for all the car park lots in the car park.

(ii)

Since for 80% of the days in the survey period, there are at least 4 occupied lots for each day, we can infer that

$P(X \geq 4) = 1 - P(X \leq 3) = 0.8$ for $X \sim \text{B}(12, p)$.

We then use GC to plot the graph involving binomial cdf and determine the x coordinate of the intersection of the curve and the line $y = 0.8$ as shown below:



Hence, the value of p is 0.412 (3 s.f.)

(iii)

Let $X \sim \mathrm{B}(12, p)$
The required conditional probability, f(p)

$$= \mathrm{P}(2 \le X < 4 \mid X \ge 1)$$

$$= \frac{\mathrm{P}(X = 2 \text{ or } X = 3)}{\mathrm{P}(X \ge 1)}$$

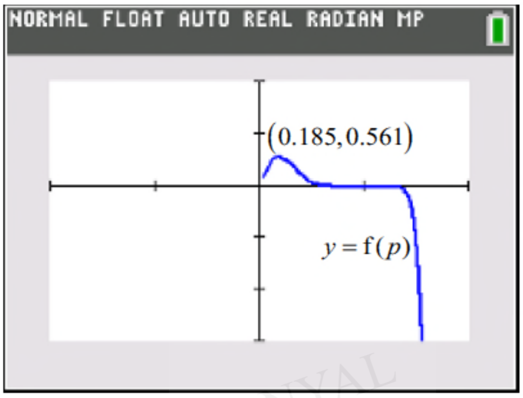
$$= \frac{\mathrm{P}(X = 2 \text{ or } X = 3)}{1 - \mathrm{P}(X = 0)}$$

$$= \frac{\binom{12}{2} p^2 (1-p)^{10} + \binom{12}{3} p^3 (1-p)^9}{1 - \left[\binom{12}{0} p^0 (1-p)^{12}\right]}$$

$$= \frac{66 p^2 (1-p)^{10} + 220 p^3 (1-p)^9}{1 - (1-p)^{12}}$$

$$= \frac{22 p^2 (1-p)^9 \left[3(1-p) + 10p\right]}{1 - (1-p)^{12}}$$

$$= \frac{22 p^2 (1-p)^9 (3 + 7p)}{1 - (1-p)^{12}}. \quad \text{(Shown)}$$



$p \approx 0.185$ give the maximum probability.