A Level H2 Math

AP and GP Test 6

Q1

(a) Find the set of values of θ lying in the interval $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ such that the sum to infinity of the geometric series $1 + \tan\theta + \tan^2\theta + \ldots$ is greater than 2. [5]

(b) The sum of the first n terms of a positive arithmetic sequence $\{u_n\}$ is given by the formula $S_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m. [4]

Q2

10 pirates live on a pirate ship and they are ranked based on their seniority.

(a) One day, the pirates found a treasure chest that consists of some gold coins. The rule which the pirates adhered by to divide all the gold coins are based on their seniority and is as follows: The most senior pirate will get 3 gold coins more than the 2nd most senior pirate. The 2nd most senior pirate will also get 3 gold coins more than the 3rd most senior pirate and so on. Thus, the most junior pirate will get the least number of gold coins.

(i) If the treasure chest contains 305 gold coins, find the number of gold coins the most senior pirate will get. [3]

(ii) Find the least number of gold coins the treasure chest must contain if all pirates get some (at least one) gold coins each. [2]

(b) The pirates need to take turns, one at a time, to be on the lookout for their ship. Each day (24 hours) is divided into 10 shifts rotated among the 10 pirates. The 1st lookout shift starts from 10pm daily and it starts with the most junior pirate to the most senior pirate. The length of their shift is also based on their seniority. The length of shift for the most senior pirate is 10% less than that of the 2nd most senior pirate. The length of shift for the 2nd most senior pirate is 10% less than that of the 3rd most senior pirate and so on. Thus, the most junior pirate has the longest shift.

(i) Show that the length of shift for the most junior pirate is 3.6848 hours, correct to 4 decimal places. [2]

(ii) Calculate the length of shift for the 6th most junior pirate. Find the start time of his shift, giving your answer to the nearest minute. [4]

Q3

The curve C has equation $y = kx^3$. The tangent at the point P on C meets the curve again at point Q. The tangent at point Q meets the curve again at point R. It is given that the x-coordinates of P, Q and R are p, q, and r respectively, where $p \neq 0$.

(i) Show that p and q satisfy the equation $\left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right) - 2 = 0$. [4]

(ii) Show that p, q and r are three consecutive terms of a geometric progression. Hence determine if this geometric series is convergent. [4]

[You may use the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ for $a, b \in \mathbb{R}$]

Answers

AP and GP Test 6

Q1

(a) For sum to infinity to exist,

$$|\tan\theta| < 1$$

$$-1 < \tan\theta < 1$$

$$-\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

Most students failed to check the range of values for $|r| < 1$ for sum to infinity to exist.

$$\frac{1}{1-\tan\theta} > 2$$

$$0 < 1-\tan\theta < \frac{1}{2}$$

$$\tan\theta > \frac{1}{2} \Rightarrow \theta > 0.464$$

Note: Many students cross multiplied to get $1 > 2(1-\tan\theta)$.
For this case it is ok as $1-\tan\theta > 0$.
In general, we should not cross multiply for inequalities unless the term multiplied is strictly positive.

Since $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$,

therefore $\{\theta \in \mathbb{R} \mid 0.464 < \theta < 0.786\}$ or $\theta : (0.464, 0.786)$

Set notation.

(b) $u_1 = S_1 = 2 \Rightarrow a = 8$

$u_2 = S_2 - S_1 = 10 \Rightarrow d = 8$

$u_{32} = a + (32-1)d = 2 + (32-1)8 = 250$

$\frac{u_{32}}{u_m} = \frac{u_m}{u_2} = \text{constant}$

u_2, u_m and u_{32} are consecutive terms of GP

$\Rightarrow (u_m)^2 = (10)(250) = 2500$

$u_m = 50$ (since it is a positive sequence)

$50 = 2 + (m-1)8 \Rightarrow m = 7$

Alternatively,

$$u_n = S_n - S_{n-1}$$
$$= 4n^2 - 2n - \left[4(n-1)^2 - 2(n-1)\right]$$
$$= 8n - 6$$

$$\frac{u_{32}}{u_m} = \frac{u_m}{u_2}$$

$$\frac{8(32)-6}{8m-6} = \frac{8m-6}{8(2)-6}$$

$$(8m-6)^2 = (250)(10) = 2500$$

$m = 7$ or $m = -5.5$ (rejected as m is a positive integer)

Q2

(a)(i)

Let a be the number of gold coins the most junior pirate will get.

$$\frac{10}{2}\left[2a+(10-1)(3)\right]=305$$

$$a=17$$

No of gold coins for most senior pirate $=17+(10-1)(3)$

$=44$

(a)(ii)

Least no of gold coins $=\frac{10}{2}\left[2(1)+(10-1)(3)\right]$

$=145$

(b)(i)

Let b be the length of shift for the most junior pirate

$$\frac{b}{1-0.9}\left(1-0.9^{10}\right)=24$$

$b=3.6848$ hr (to 4 d.p.) (shown)

(b)(ii)

Length of shift for 6th most junior pirate $=3.6848(0.9)^5$

$=2.18$ hr

Length of 1st 5 shifts $=\frac{3.6848}{1-0.9}\left(1-0.9^5\right)$

$=15.090$

$=15$ hrs 5 mins

Start time of shift $=1.05$pm

Q3

(i) $y = kx^3$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 3kx^2$ Point $P = (p,\ kp^3)$, Point $Q = (q,\ kq^3)$, Point $R = (r,\ kr^3)$	
Equation of tangent at point *P*: $y - kp^3 = 3kp^2(x - p)$ When tangent meets the curve again at *Q*: $kq^3 - kp^3 = 3kp^2(q - p)$ $q^3 - p^3 = 3p^2(q - p)$ $(q-p)(q^2 + pq + p^2) = 3p^2(q - p)$ $(q-p)(q^2 + pq - 2p^2) = 0$ $q^2 + pq - 2p^2 = 0$ since $p \neq q$ because *P* and *Q* are different points Dividing by p^2 : $\left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right) - 2 = 0$ (Shown)	Note that the gradient to tangent at point *P* is not $3kx^2$. You need to substitute *x* by *p* in $\frac{\mathrm{d}y}{\mathrm{d}x} = 3kx^2$ to get the gradient of tangent at point *P*.

(ii) $\left(\frac{q}{p}\right)^2+\left(\frac{q}{p}\right)-2=0$ $\Rightarrow \left(\frac{q}{p}+2\right)\left(\frac{q}{p}-1\right)=0$ $\Rightarrow \frac{q}{p}=-2$ or $\frac{q}{p}=1$ (rejected since $q\neq p$) Similarly for the other case, $\frac{r}{q}=-2$ $\therefore \frac{q}{p}=\frac{r}{q}=-2$ Since the common ratio is the same, p, q and r are three consecutive terms of a geometric progression. As $\lvert\text{common ratio}\rvert=2>1$, the geometric series is not convergent.	

Marker's comments

For part (i), while many students are able to find the equation of tangent, most students who had found the equation of tangent at P did not know how to continue from there. They need to observe more carefully what other information is given on the tangent to continue. In this case it is the fact that the tangent line cuts the curve again at point Q. This will lead to substituting x by q in the equation of tangent.

For part (ii), students must recall the condition for a sequence to be a GP, in this case $\frac{q}{p}=\frac{r}{q}$, and work towards it. As for the second part, students must be aware that the condition for geometric series to be convergent is $\lvert\text{common ratio}\rvert<1$.