A Level H2 Math

AP and GP Test 5

Q1

A farmer owns a plot of farmland. To prepare for wheat planting, the farmer has to plough the farmland before sowing wheat seeds. At the start of the first week, 300 m^2 of the farmland is ploughed. The farmer ploughs another 100 m^2 of the farmland at the beginning of each subsequent week. To sow wheat seeds, the farmer is considering two different options.

(a) In the first option, the farmer sows wheat seeds on 60% of the **unsown** ploughed land at the end of each week.

(i) Find the area of **unsown** ploughed land at the end of the second week. [1]

(ii) Show that the area of **unsown** ploughed land at the end of the nth week is given by

$$\left[0.4^n(300)+k\left(1-0.4^{n-1}\right)\right]\text{ m}^2,$$

where k is an exact constant to be determined. [3]

(iii) Find the number of complete weeks required for the area of **unsown** ploughed land to first fall below 70 m^2. [3]

(b) In the second option, the farmer sows 80 m^2 of the **unsown** ploughed land at the end of the first week. At the end of each subsequent week, he sows 20 m^2 of the **unsown** ploughed land more than in the previous week. This means that the area of sown ploughed land is 100 m^2 in the second week, 120 m^2 in the third week, and so on.

(i) Find, in terms of n, the area of **unsown** ploughed land at the end of the nth week. [4]

(ii) Find the number of complete weeks required for the farmer to finish sowing all the ploughed farmland in this option. Deduce the area of ploughed land to be sown in the final week. [4]

Q2

In a training session, an athlete runs from a starting point S towards his coach in a straight line as shown in the diagrams below. When he reaches the coach, he runs back to S along the same straight line. A lap is completed when he returns to S. At the beginning of the training session, the coach stands at A_1 which is 30 m away from S. After the first lap, the coach moves from A_1 to A_2 and after the second lap, he moves from A_2 to A_3 and so on. The distance between A_i to A_{i+1} is denoted by A_iA_{i+1}, $i \in \mathbb{Z}^+$.

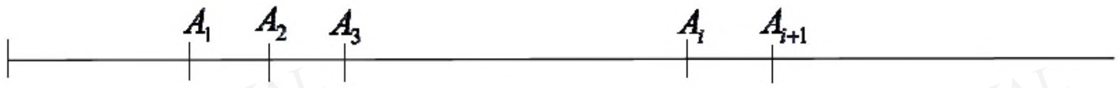

Figure 1

(i) For training regime 1 (shown in Figure 1), the coach ensures that the distance $A_iA_{i+1} = 3$ m for $i \in \mathbb{Z}^+$. Find the least number of laps that the athlete must complete so that he covers a total distance of more than 3000 m. **[3]**

Figure 2

(ii) For training regime 2 (shown in Figure 2), after the first lap, the coach ensures that the distances $A_1A_2 = 2$ m, $A_2A_3 = 6$ m and the distance $A_{i+1}A_{i+2} = 3A_iA_{i+1}$ where $i \in \mathbb{Z}^+$. Show that the distance the coach is away from S **just before** the athlete completed r laps is $\left(3^{r-1} + 29\right)$ m.

Hence find the distance run by the athlete after n complete laps. Also find how far the athlete is from the coach after he has run 8 km. **[6]**

Q3

A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero and $a>0$. The first three terms of the geometric series are equal to the first, eighth and thirteenth terms respectively of the arithmetic series.

(i) Show that $7r^2-12r+5=0$. [2]

(ii) Deduce that the geometric series is convergent. [2]

(iii) The sum of the first n terms of the geometric series is denoted by S_n. Find the smallest value of n for S_n to be within 0.1% of the sum to infinity of the geometric series. [4]

(iv) Find exactly the sum of the first 2017 terms of the arithmetic series, leaving your answer in terms of a. [3]

Answers

AP and GP Test 5

Q1

(a)(i)
Area of **unsown** ploughed land
$=0.4\left[0.4(300)+100\right]$

$=88\text{ m}^2$

(a)(ii)

n	Beginning of week	End of week
1	300	$0.4(300)$
2	$0.4(300)+100$	$0.4\left[0.4(300)+100\right]$ $=0.4^2(300)+0.4(100)$
3	$0.4^2(300)+0.4(100)$ $+100$	$0.4\left[0.4^2(300)+0.4(100)+100\right]$ $=0.4^3(300)+0.4^2(100)+0.4(100)$
..	…	…
n	…	$0.4^n(300)+0.4^{n-1}(100)+...$ $+0.4^2(100)+0.4^1(100)$

Area of land **unsown** ploughed land at the end of nth week

$$=0.4^n(300)+100\left[\frac{0.4\left(1-0.4^{n-1}\right)}{1-0.4}\right]$$

$$=\left[0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)\right]\text{ m}^2$$

$\therefore$ the value of k is $\frac{200}{3}$.

(a)(iii)
Method 1

$$0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)<70$$

$$0.4^n(300)+\frac{200}{3}-\frac{200}{3}(0.4)^{-1}0.4^n<70$$

$$\frac{400}{3}\left(0.4^n\right)<\frac{10}{3}$$

$$0.4^n<\frac{1}{40}$$

$$n>\ln\left(\tfrac{1}{40}\right)\Big/\ln 0.4$$

$$n>4.02588$$

Hence the number of complete weeks required is 5.

Method 2

$$0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)<70$$

Using GC,
when $n = 4$, unsown ploughed land = 70.08 (> 70)
when $n = 5$, unsown ploughed land = 68.032 (< 70)
when $n = 6$, unsown ploughed land = 67.213 (< 70)

Hence the number of complete weeks required is 5.

(b)(i)

n	Beginning of week	End of week
1	300	$300-80$
2	$300+(100)-80$	$300+(100)-80-100$
3	$300+2(100)-80-100$	$300+2(100)-80-100-120$
..	$\cdots$	$\cdots$
n	$\cdots$	$300+(n-1)(100)-80-100$ $-\cdots-[80+20(n-1)]$

Area of **unsown** ploughed land at the end of nth week

$$=300+100(n-1)-\frac{n}{2}[2(80)+20(n-1)]$$

$$=300+100n-100-\frac{n}{2}(140+20n)$$

$$=300+100n-100-70n-10n^2$$

$$=-10n^2+30n+200$$

(b)(ii)
For the farmer to finish sowing all the ploughed farmland,
$-10n^2+30n+200\le 0$

Method 1:
Solving the inequality,
$n\ge 6.21699$ or $n\le -3.21699$ (rejected)
Hence the number of complete weeks is 7.

Method 2:
Using GC to set up a table,
When $n=6$, area unsown $=20$
When $n=7$, area unsown $=-80$
When $n=8$, area unsown $=-200$
Hence the number of complete weeks is 7.

In week 6, the area of **unsown** ploughed land

$= -10(6)^2 + 30(6) + 200 = 20 \text{ m}^2$

$\therefore$ area of ploughed land to be **sown** in week 7 (the final week)

$= 20 + 100 = 120 \text{ m}^2$

Q2

(i)

Distance travelled per lap is in AP:

$a = 2(30) = 60$, $d = 2 \times 3 = 6$.

Given total distance travelled > 3000

$\frac{n}{2}[\,2(60) + (n-1)6] > 3000$

$3n^2 + 57n - 3000 > 0$

$(n + 42.52)(n - 23.52) > 0$

$n < -42.52$ or $n > 23.52$

Since $n \in \mathbb{Z}^+$, least $n = 24$

(ii)

Distance of the coach from S just before the runner completes the rth lap

$= 30 + 2(3^0) + 2(3^1) + 2(3^2) + \ldots + 2(3^{r-2})$

$= 30 + 2\left(1 + 3 + 3^2 + \ldots. + 3^{r-2}\right)$

$= 30 + 2\left(\frac{3^{r-1} - 1}{3 - 1}\right)$

$= 30 + (3^{r-1} - 1)$

$= 3^{r-1} + 29$

Distance covered by the athlete after n laps

$= \sum_{r=1}^{n} 2\left(3^{r-1} + 29\right)$

$= 2\sum_{r=1}^{n} 3^{r-1} + \sum_{r=1}^{n}(58)$

$= 2\sum_{r=1}^{n} 3^{r-1} + 58n$

$= 2\left(\frac{3^n - 1}{3 - 1}\right) + 58n$

$= \left(3^n - 1\right) + 58n$

When $D = 8000$m

$8000 = \left(3^n - 1\right) + 58n$

From GC,

$n = 8.1254$

Hence the athlete has run 8 complete laps.

The athlete has completed 7024 m

Hence he still have 8000-7024=976 m

On the 9th lap, the coach is $3^{9-1} + 29 = 6590$ m from S.

Hence the athlete would be 6590- 976 = 5614 m away from the coach once he finishes 8 km.

Q3

i

$$ar = a + (8-1)d \Rightarrow d = \frac{ar-a}{7}$$

$$ar^2 = a + (13-1)d \Rightarrow d = \frac{ar^2-a}{12}$$

$$\frac{ar-a}{7} = \frac{ar^2-a}{12}$$

$$12r - 12 = 7r^2 - 7$$

$$7r^2 - 12r + 5 = 0$$

ii

From the GC, $r = \frac{5}{7}$ or $r = 1$.

Since $d \neq 0$, the terms of the geometric series are distinct we conclude that $r \neq 1$. Hence, $r = \frac{5}{7}$.

As $|r| = \left|\frac{5}{7}\right| < 1$, the geometric series is convergent.

iii

$$|S_\infty - S_n| < 0.001 S_\infty$$

$$\left| \frac{a}{1-\frac{5}{7}} - \frac{a\left(1-\left(\frac{5}{7}\right)^n\right)}{1-\frac{5}{7}} \right| < 0.001\left(\frac{a}{1-\frac{5}{7}}\right)$$

$$\left|\frac{a}{1-\frac{5}{7}}\right| \left|1 - \left(1-\left(\frac{5}{7}\right)^n\right)\right| < 0.001\left(\frac{a}{1-\frac{5}{7}}\right)$$

$$\left(\frac{5}{7}\right)^n < 0.001 \quad (\because a > 0)$$

$$n \ln\left(\frac{5}{7}\right) < \ln 0.001$$

$$n > \frac{\ln 0.001}{\ln \frac{5}{7}}$$

$$n > 20.53$$

Smallest value of n is 21.

iv

$$d = \frac{ar-a}{7} = \frac{a\left(\frac{5}{7}\right)-a}{7} = -\frac{2}{49}a$$

The sum of the first 2017 terms of the arithmetic series

$$= \frac{2017}{2}\left[2a + (2017-1)\left(-\frac{2}{49}a\right)\right]$$

$$= -\frac{566777}{7}a$$