A Level H2 Math

AP and GP Test 4

Q1

A fund is started at \$6000 and compound interest of 3% is added to the fund at the end of each year. If withdrawals of \$$k$ are made at the beginning of each of the subsequent years, show that the amount in the fund at the beginning of the $(n+1)$th year is

$$\$\frac{100}{3}\left[(180-k)(1.03)^n+k\right].$$ [5]

(i) It is given that $k = 400$. At the beginning of which year, for the first time, will the amount in the fund be less than \$1000? [2]

(ii) If the fund is fully withdrawn at the beginning of sixteenth year, find the least value of k to the nearest integer. [2]

Q2

There are 25 toll stations, represented by T_1, T_2, T_3,……, T_{25} along a 2000 km stretch of highway. T_1 is located at the start of the highway and T_2 is located x km from T_1. Subsequently, the distance between two consecutive toll stations is 2 km more than the previous distance. Find the range of values x can take. [3]

Use $x = 60$ for the rest of this question.

Each toll station charges a fee based on the distance travelled from the previous toll station. The fee structure at each toll station is as follows:
For the first 60 km, the fee per km will be 5 cents. For every additional 2 km, the fee per km will be 2% less than the previous fee per km.

(i) Find, in terms of n, the amount of fees a driver will need to pay at T_n. [3]

(ii) Find the total amount of fees a driver will need to pay, if he drives from T_1 to T_n. Leave your answer in terms of n. [4]

More toll stations are built along the highway in the same manner, represented by T_{26}, T_{27}, T_{28},…….. beyond the 2000 km stretch.

(iii) If a driver starts driving from T_1 and only has \$200, at which toll station will he not have sufficient money for the fees? [2]

Q3

(a) The fifth, ninth and eleventh terms of a geometric progression are also the seventh, twenty-fifth and forty-ninth terms of an arithmetic progression with a non-zero common difference respectively.
Show that $3R^6 - 7R^4 + 4 = 0$, where R is the common ratio of the geometric progression and determine if the geometric progression is convergent. [4]

(b) A semicircle with radius 12 cm is cut into 8 sectors whose areas follow a geometric progression. The first sector, which is the largest, has an area of A cm^2. The second sector has an area of Ar cm^2, the third sector has an area of Ar^2 cm^2, and so on, where r is a positive constant. Given also that the total area of the odd-numbered sectors is 10π cm^2 more than that of the even-numbered sectors, find the values of A and r. [5]

(c) The production levels of a particular coal mine in any year is 4% less than in the previous year. Show that the total production of the coal mine can never exceed 25 times the production in the first year. [2]

Answers

AP and GP Test 4

Q1

Yr	Amount at the beginning of yr	Amount at the end of yr
1	6000	$6000(1.03)$
2	$6000(1.03)-k$	$[6000(1.03)-k](1.03)$ $=6000(1.03)^2-k(1.03)$
3	$6000(1.03)^2-k(1.03)-k$ $=6000(1.03)^2-k(1.03)-k$	$\left[6000(1.03)^2-k(1.03)-k\right](1.03)$ $=6000(1.03)^3-k(1.03)^2-k(1.03)$

By inspection, amount in the fund at the end of nth year

$$= 6000(1.03)^n - k(1.03)^{n-1} - k(1.03)^{n-2} - \ldots - k(1.03)$$

Amount in the fund at the beginning of $(n+1)$th year

$$= 6000(1.03)^n - k(1.03)^{n-1} - k(1.03)^{n-2} - \ldots - k(1.03) - k$$

$$= 6000(1.03)^n - k\left[1 + 1.03 + (1.03)^2 + \cdots + (1.03)^{n-1}\right]$$

$$= 6000(1.03)^n - k\left\{\frac{1\left[1-(1.03)^n\right]}{1-1.03}\right\}$$

$$= 6000(1.03)^n + \frac{100}{3}k\left[1-(1.03)^n\right]$$

$$= \frac{100}{3}\left[180(1.03)^n + k - k(1.03)^n\right]$$

$$= \frac{100}{3}\left[(180-k)(1.03)^n + k\right] \quad \text{[Shown]}$$

(i) Given $k = 400$,

$$\frac{100}{3}\left[(180-400)(1.03)^n + 400\right] < 1000$$

$$-220(1.03)^n + 400 < 30$$

$$(1.03)^n > \frac{37}{22} \text{ (or 1.6818)}$$

$$n \ln 1.03 > \ln\frac{37}{22}$$

$$n > \frac{\ln\frac{37}{22}}{\ln 1.03} = 17.6 \text{ (3 sf)}$$

Least $n = 18$

Or: use GC, table of values gives

$$\text{least } n = 18$$
$$n+1 = 19$$

Therefore, at the beginning of 19th year, the amount in the fund will be less than \$1000 for the first time

(ii) When $n+1=16 \Rightarrow n=15$,

$$\frac{100}{3}\left[(180-k)(1.03)^{15}+k\right] \le 0$$
$$(180-k)(1.03)^{15}+k \le 0$$
$$180(1.03)^{15}+k\left[1-(1.03)^{15}\right] \le 0$$
$$k\left[1-(1.03)^{15}\right] \le -180(1.03)^{15}$$
$$k\left[(1.03)^{15}-1\right] \ge 180(1.03)^{15}$$
$$k \ge \frac{180(1.03)^{15}}{(1.03)^{15}-1}$$
$$k \ge 502.6$$
$$\text{Least } k = \underline{503} \text{ (nearest integer)}$$

Or: from GC (plot graph or table of values),

$$\text{least } k = \underline{503} \text{ (nearest integer)}$$

Q2

$$x+(x+2)+(x+2(2)) + \dots +(x+23(2)) \le 2000$$

This is an AP with first term= x, common difference = 2, number of terms = 24

$$\frac{24}{2}[2x+23(2)] \le 2000$$

$$0 < x \le \frac{181}{3}$$

10(i)

n	Amount paid at T_n
2	$60(0.05)$
3	$60(0.05)+2(0.05)(0.98)$
4	$60(0.05)+2(0.05)(0.98)+2(0.05)(0.98)^2$
	. .
n	$60(0.05)+2(0.05)(0.98)+2(0.05)(0.98)^2+\ldots+2(0.05)(0.98)^{n-2}$

Amount of fees at T_n $= 3+\frac{0.098(1-0.98^{n-2})}{1-0.98}$

$$=3+4.9(1-0.98^{n-2})$$

$$=7.9-4.9(0.98^{n-2})$$

ii

$$\sum_{r=2}^{n}\left[7.9-4.9(0.98^{r-2})\right]$$

$$=\sum_{r=2}^{n}7.9-4.9\sum_{r=2}^{n}(0.98^{r-2})$$

$$=7.9(n-1)-4.9\left[\frac{1(1-0.98^{n-1})}{1-0.98}\right]$$

$$=7.9(n-1)-245(1-0.98^{n-1})$$

$$=7.9n+245(0.98^{n-1})-252.9$$

iii

Let $f(n)=7.9n+245(0.98^{n-1})-252.9$. Note that $f(n)$ is increasing in n

Consider $7.9n+245(0.98^{n-1})-252.9>200$

44	197.47
45	203.32
46	209.21

Using GC, $n \ge 45$

He will not have sufficient money at the 45th toll station.

Q3

(a) Let a denote the first term of the geometric progression.
Likewise, let b and d denote the first term and common difference of the arithmetic progression.

$\therefore$ $ar^4 = b + 6d$ ...Eq(1)
$ar^8 = b + 24d$ ...Eq(2)
$ar^{10} = b + 48d$ ...Eq(3)

Eq(2) – Eq(1): $ar^8 - ar^4 = 18d$ ...Eq(4)
Eq(3) – Eq(2): $ar^{10} - ar^8 = 24d$ ...Eq(5)

Eq(5)/Eq(4): $\dfrac{ar^8(r^2-1)}{ar^4(r^4-1)} = \dfrac{24d}{18d}$

$$\frac{r^4}{r^2+1} = \frac{4}{3}$$

$3r^4 = 4r^2 + 4$ (Shown)

From GC, $r = \pm\sqrt{2}$ so $|r| > 1$
Hence, the geometric progression is not convergent.

(b)
Let a be the 1st term and r be the common ratio of the G.P.

$$S_8 = \frac{A(1-r^8)}{1-r} = 72\pi \quad \text{----- (1)}$$

$$S_{odd} - S_{even} = 10\pi$$

$$\Rightarrow \frac{A(1-(r^2)^4)}{1-r^2} - \frac{Ar(1-(r^2)^4)}{1-r^2} = 10\pi$$

$$\frac{A(1-r^8)}{(1-r)(1+r)}[1-r] = 10\pi \quad \text{----- (2)}$$

(1) ÷ (2):

$$\frac{1-r}{1+r} = \frac{10}{72}$$

$$72 - 72r = 10 + 10r$$

$$82r = 62$$

$$r = 0.75610$$

Substituting into equation (1), $A = 61.8$ (to 3 s.f.)
Let the production level in the first year be a.

Total production of the coal mine $= \dfrac{a}{1-0.96} = 25a$

Thus, the total production of the coal mine can never exceed 25 times the production in the first year.