A Level H2 Math

AP and GP Test 2

Q1

An arithmetic sequence $u_1, u_2, u_3, \ldots$ is such that the difference between the fourteenth term and the fifth term is equal to the sum of the terms between the fifth term and the fourteenth term (both inclusive). Given further that that sum of the third, fifth and fourteenth terms is 19, find the common difference of the sequence. [4]

Hence, or otherwise, find the largest of value of n such that the sum of the first n terms is positive. [2]

Q2

Kumar wishes to purchase a gift priced at \$280 for his mother.

Starting from January 2017,

- Kumar saves \$100 in his piggy bank on the 1st day of each month;
- Kumar donates 30% of his money in his piggy bank to charity on the 15th day of each month and
- Kumar's father puts an additional \$20 in Kumar's piggy bank on the 25th day of each month.

(i) Find the amount of money in Kumar's piggy bank at the end of March 2017. [2]

(ii) Show that the amount of money in Kumar's piggy bank at the end of n months is

$$300\left(1-0.7^n\right).$$ [3]

(iii) At the end of which month will Kumar first be able to purchase the gift for his mother? [2]

Q3

A researcher is investigating the elasticity of a new material. In the experiment, he stretched an extensible string of length 30 cm using a machine.

Each stretch is followed by a contraction. The initial stretch leads to an elongation of 10 cm and is followed by a contraction of 0.1 cm. The elongation resulting from each subsequent stretch is $\frac{10}{11}$ of the elongation caused by the previous stretch. Each subsequent contraction is 0.001 cm less than the previous contraction.

(i) Show that the length of the string after two stretches is 48.892 cm correct to 3 decimal places. [2]

(ii) Find the length of the string after it has been stretched n times, in terms of n. [3]

(iii) The string loses its elasticity completely when contraction exceeds elongation in a stretch. Find the minimum number of stretches for the string to lose its elasticity. [2]

(iv) The researcher coats a new string of the same initial length with another material. Now the string does not contract after every stretch while its elongation properties remain unchanged. Justify why it is impossible for the string to be elongated beyond 140 cm. [1]

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Answers

AP and GP Test 2

Q1

Let the first term be a and the common difference be d.

$$\sum_{k=5}^{14} u_k = |u_{14} - u_5|$$

$$S_{14} - S_4 = |(a+13d) - (a+4d)|$$

$$\frac{14}{2}(2a+13d) - \frac{4}{2}(2a+3d) = |9d|$$

$$14a + 91d - 4a - 6d = |9d|$$

$$10a + 85d = |9d|$$

$$10a + 85d = 9d \quad \text{or} \quad 10a + 85d = -9d$$

$$5a + 38d = 0 \text{---- (1)} \quad \text{or} \quad 5a + 47d = 0 \text{---- (1)}$$

$$u_3 + u_5 + u_{14} = 19$$

$$(a+2d) + (a+4d) + (a+13d) = 19$$

$$3a + 19d = 19 \text{ ----- (2)}$$

Solving simultaneously, from GC,

$a = 38,\ d = -5$ or $a = \frac{893}{46},\ d = -\frac{95}{46}$

Hence the common difference is -5 or $-\frac{95}{46}$

$$S_n > 0 \qquad\qquad S_n > 0$$

$$\frac{n}{2}(2a + (n-1)d) > 0 \quad \text{or} \quad \frac{n}{2}(2a + (n-1)d) > 0$$

$$n(81 - 5n) > 0 \qquad\qquad n\left(\frac{1881}{92} - \frac{95}{92}n\right) > 0$$

$$0 < n < \frac{81}{5} = 16.2 \quad \text{or} \quad 0 < n < \frac{99}{5} = 19.8$$

Hence, the largest value of n is 16 or 19.

Q2

(i)

		Amount of $ Kumar has @ the …		
		Beginning	Middle	End
Jan 2017	1	100	$0.7(100)$	$0.7(100)+20$
Feb 2017	2	$100+0.7(100)+20$	$0.7[100+0.7(100)+20]$	$0.7(100)+0.7^2(100)$ $+0.7(20)+20$
Mar 2017	3	$100+0.7(100)$ $+0.7^2(100)$ $+0.7(20)+20$	$0.7[100+0.7(100)$ $+0.7^2(100)+0.7(20)+20]$	$0.7(100)+0.7^2(100)$ $+0.7^3(100)+0.7^2(20)$ $+0.7(20)+20$
		…	…	…
	n			$0.7(100)+0.7^2(100)$ $+0.7^3(100)+\ldots+0.7^n(100)$ $+0.7^{n-1}(20)+\ldots+0.7^2(20)$ $+0.7(20)+20$

Amount of money Kumar has at the end of March 2015

$$= \$\left[0.7(100)+0.7^2(100)+0.7^3(100)+0.7^2(20)+0.7(20)+20\right]$$

$$= \$197.10$$

Most students were able to get the value correct.

(ii) Amount of money Kumar has at the end of n months

$$= 0.7(100)+0.7^2(100)+0.7^3(100)+\ldots+0.7^n(100)$$
$$+0.7^{n-1}(20)+\ldots+0.7^2(20)+0.7(20)+20$$

$$= 100\left(0.7+0.7^2+\ldots+0.7^n\right)+20\left(1+0.7+0.7^2+\ldots+0.7^{n-1}\right)$$

$$= 100\left[\frac{0.7\left(1-0.7^n\right)}{1-0.7}\right]+20\left[\frac{1-0.7^n}{1-0.7}\right]$$

$$= \frac{700}{3}\left(1-0.7^n\right)+\frac{200}{3}\left(1-0.7^n\right)$$

$$= 300\left(1-0.7^n\right) \text{ (shown)}$$

There were a many methods presented by students. However, because it is a show question, their working must be clear. Credit were not given to students who just state that $a = 90, r = 0.7$ unless the explanation on why $a = 90$ is clear.

(iii)

$$300\left(1-0.7^n\right) \geq 280$$

$$1-0.7^n \geq \frac{14}{15}$$

$$0.7^n \leq \frac{1}{15}$$

$$n \geq 7.59$$

Kumar will first be able to purchase the gift for his mother at the 8th month. (or August 2017)

Quite badly done by students wo did not use the GC table method with many students not realized that $\ln 0.7 < 0$ and hence there is a need to change the inequality sign when dividing by $\ln 0.7$ on both sides of the unequalities.

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Q3

(i)

Stretch count, n	Length of string before stretch	Elongation after stretch, u_n	Contraction after stretch, t_n	Final length of string
1	30	10	0.1	$30+10-0.1=39.9$
2	39.9	$10\left(\frac{10}{11}\right)$	$0.1-0.001$ $=0.099$	$39.9+10\left(\frac{10}{11}\right)-0.099$ $=48.8919$

$$30+10-0.1+10\left(\frac{10}{11}\right)-0.099$$

$$=48.8919$$

$$=48.892 \ (3 \text{ dp})$$

(ii)

Total length of string

$$=30+u_1+u_2+\ldots+u_n-\left(t_1+t_2+\ldots+t_n\right)$$

$$=30+10+10\left(\frac{10}{11}\right)+\ldots+10\left(\frac{10}{11}\right)^{n-1}$$

$$-\left(0.1+\left(0.1-0.001(1)\right)+\ldots+\left(0.1-0.001(n)\right)\right)$$

$$\sum_{i=1}^{n}u_i=\frac{10\left[1-\left(\frac{10}{11}\right)^n\right]}{1-\frac{10}{11}}=110\left(1-\left(\frac{10}{11}\right)^n\right)$$

$$\sum_{i=1}^{n}t_i=\frac{n}{2}\left[2(0.1)+(n-1)(-0.001)\right]=\frac{n}{2000}(201-n)$$

Length of string after n stretches

$$=30+110\left(1-\left(\frac{10}{11}\right)^n\right)-\frac{n}{2000}(201-n)$$

(iii)

$t_n > u_n$

$$0.1+(n-1)(-0.001)>(10)\left(\frac{10}{11}\right)^{n-1}$$

$$0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}>0$$

Using GC,

when $n=58, 0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}=-7.1364\times10^{-4}$

when $n=59, 0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}=0.00226$

Therefore, the minimum number of stretches is 59.

(iv)

$$S_\infty = 30+\frac{10}{1-\frac{10}{11}}=140 \quad (\text{since } 0<r<1)$$

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Since the sum to infinity, S_∞ is 140, it is impossible for the string to be stretched beyond 140cm.

OR

The theoretical maximum is 140 cm so it is impossible for the strong to be stretched beyond 140 cm.